Generating family homologies of Legendrian submanifolds and moduli spaces of gradient staircases

Introduction

Legendrian submanifolds are rigid submanifolds of contact manifolds.
Despite being of large codimensions in contact manifolds, homotopy theory does not suffice to understand their isotopies.

Theorem. (Folklore)
Smooth isotopy classes of Legendrian submanifolds split into infinitely many distinct Legendrian isotopy classes.

Three approaches to study the contact rigidity of Legendrian submanifolds:
- Pseudo-holomorphic curves theory;
- Generating families;
- Microlocal sheaf theory.

These techniques are conjectured to produce "equivalent" invariants.
But generating family invariants currently lack of algebraic structure.

Generating family invariants are constructed from Morse moduli spaces $\mathcal{M}(c_-, c_+; g)$.
The relevant function to consider is called the difference function and is defined on $B \times \mathbb{R}^{2N}$.
Via the mean of generating families, the Legendrian submanifold embeds in $B \times \mathbb{R}^{2N}$.

Henry-Rutherford combinatorial dg-algebra for Legendrian knots:
- Constructed from moduli spaces of chord paths $\mathcal{M}^c(c_-, c_+)$;
- Geometrical grounding in generating families;
- Moduli spaces of gradient staircases $\mathcal{M}^{st}(c_-, c_+)$ are a bridge between $\mathcal{M}^c(c_-, c_+)$ and $\mathcal{M}(c_-, c_+; g)$.

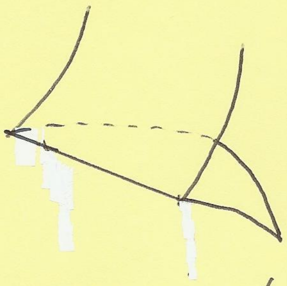
Theorem. (Henry-Rutherford, 2013)
There exists a one-to-one correspondence between $\mathcal{M}^c(c_-, c_+)$ and $\mathcal{M}^{st}(c_-, c_+)$, provided they are finite.

Conjecture. (Henry-Rutherford, 2013)
If $s \in [0,1]$ is small enough and $g_s = (s^{-1} g_{\mathbb{R}}) \oplus g_F$, then there exists a one-to-one correspondence between $\mathcal{M}^{st}(c_-, c_+)$ and $\mathcal{M}(c_-, c_+; g_s)$, provided they are finite.

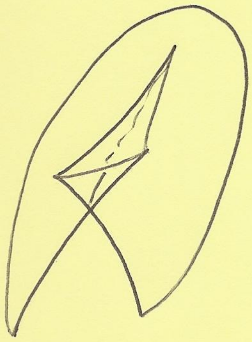
It is difficult to define $\mathcal{M}^c(c_-, c_+)$ for higher-dimensional Legendrian submanifolds.
Asking whether $\mathcal{M}^{st}(c_-, c_+)$ and $\mathcal{M}(c_-, c_+; g_s)$ are in one-to-one correspondence still makes sense in higher dimensions.
This question is adressed by a compactness-gluing strategy for the adiabatic limit $s \to 0$.

For the compactness theorem, let us define $\overline{\mathcal{M}^{st}}(c_-, c_+) = \bigcup \mathcal{M}^{st}(c_-, c_1) \times \mathcal{M}^{st}(c_1, c_2) \times \dots \times \mathcal{M}^{st}(c_k, c_+)$.
It is the analogue for moduli spaces of gradient staircases of moduli spaces of broken gradient trajectories in Morse theory.

<u>Theorem.</u> (F.)
If $\Lambda \to B$ has only Whitney pleat singularities and Λ is generic, then for all $\gamma_s \in \mathcal{M}(c_-, c_+; g_s)$ with $g_s = (s^{-1} g_B) \oplus g_F$ and $s \to 0$, there exists a subsequence $s_k \to 0$ and $e \in \overline{\mathcal{M}^{st}}(c_-, c_+)$ such that $\gamma_{s_k} \xrightarrow[k \to +\infty]{} e$ in the Floer-Gromov topology.



Whitney pleat ✓



Swallowtail X

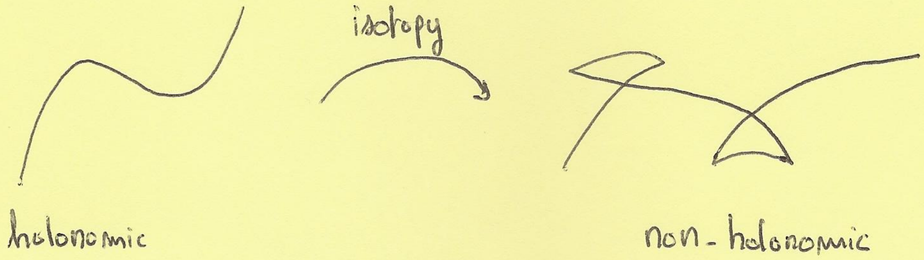
Need to explain:
- Genericity assumption;
- Floer-Gromov topology.

When $\dim \Lambda = 1$, the assumption on the singularities is empty.
When $\dim \Lambda \geq 2$, the assumption on the singularities is equivalent to a homological condition (Alvarez-Gavela, 2016)

I. Henry-Rutherford limiting process and gradient staircases

1. Generating family homologies

Holonomic sections of J^1B are Legendrian submanifolds of (J^1B, ξ_B).
They are all Legendrian isotopic to the zero section 0_B of J^1B.
Not all Legendrian submanifolds of (J^1B, ξ_B) which are Legendrian isotopic to 0_B are holonomic.
But they are obtained by graph reduction of an holonomic section of $(J^1(B\times\mathbb{R}^{2N}), \xi_{B\times\mathbb{R}^{2N}})$.



Definition. (Folklore)
A generating family of a Legendrian submanifold Λ of (J^1B, ξ_B) is a smooth map $f: B\times\mathbb{R}^{2N}_\eta \to \mathbb{R}$ such that:
1. The fibrewise critical set $\Sigma_f = \partial_\eta f^{-1}(0)$ is transversally cut-out.
2. $\Lambda = \{(b, \partial_b f(b,\eta), f(b,\eta)) ; (b,\eta)\in\Sigma_f\} = \Lambda_f$

Remark.
$C = T^*B\times\{0_{\mathbb{R}^{2N}}\}$ is a coisotropic submanifold of $T^*(B\times\mathbb{R}^{2N})$.
$T^*(B\times\mathbb{R}^{2N})/C^\perp$ is canonically symplectomorphic to T^*B.
Γ_{df} is a Lagrangian submanifold of $T^*(B\times\mathbb{R}^{2N})$ which is transverse to C.
$L_f = \Gamma_{df}/C^\perp$ is an immersed Lagrangian submanifold of T^*B.
L_f lifts through the values of f to an immersed Legendrian submanifold of J^1B.

When a Legendrian submanifold admits a generating family, it admits infinitely many others.
But they should be treated as equal.

Definition. (Folklore)
- Stabilisation move. $f\oplus Q(b,\eta_1,\eta_2) = f(b,\eta_1)+Q(\eta_2)$, Q non-degenerate quadratic form.
- Fibre-preserving diffeomorphism. $f_\varphi(b,\eta) = f(\varphi(b,\eta))$, $\varphi: B\times\mathbb{R}^{2N}\circlearrowleft$ fibre-preserving diffeomorphism.

Two generating families are equivalent whenever they can be made equal by applying a finite number of these moves.

Remark.
Two equivalent generating families generate the same Legendrian submanifold.

Generating families are suited to study isotopies of Legendrian submanifolds.

Theorem. (Chaperon (1984), Laudenbach-Sikorav (1985), Sikorav (1986), Chekanov (1996))

If $(\Lambda_t)_{t\in[0,1]}$ is a Legendrian isotopy, then any generating family of Λ_0 (provided there is any) gives rise to an equivalent generating family of Λ_1.

Difference function: $\delta(b,\eta_1,\eta_2) = f_1(b,\eta_1) - f_2(b,\eta_2)$.

Proposition. (Folklore)

The critical points of δ are of two types:

1. The positively/negatively valued critical points of δ are in one-to-one correspondence with the Reeb chords of Λ.
2. If $\varepsilon>0$ is small enough, then δ is Morse-Bott in $\{-\varepsilon<\delta<\varepsilon\}$.

It has a unique critical submanifold in $\{-\varepsilon<\delta<\varepsilon\}$ and it is diffeomorphic to Λ.

Idea. Use Morse theory of the difference function to construct invariants of Legendrian submanifolds.

Morse theory is ill-behaved on non-compact manifolds.
Without any assumption on their behaviour at infinity, generating families are unlikely to produce interesting invariants.
All the generating families will be linear-at-infinity.

Definition. (Traynor, 2001)

Let $\omega>\varepsilon>0$ such that all the positive critical values of δ are contained in (ε,ω).
Generating family homology of (f_1,f_2): $GFH_\cdot(f_1,f_2) = H_{\cdot+N+1}(\{\delta<\omega\},\{\delta<\varepsilon\};\mathbb{F}_2)$.

Theorem. (Traynor-Sabloff, 2013, F.)

If $f_1\sim f_2$, then there exist $N_1,N_2\in\mathbb{Z}$ such that for all f:

$$GFH_\cdot(f_1)\simeq GFH_\cdot(f_2),$$
$$GFH_\cdot(f_1,f)\simeq GFH_{\cdot+N_1}(f_2,f),$$
$$GFH_\cdot(f,f_1)\simeq GFH_{\cdot+N_2}(f,f_2),$$

as graded $\mathbb{F}_2$-vector spaces.

GFH is hard to compute:

- Morse homology is an efficient homotopy invariant of closed manifolds.

Differential of the Morse chain complex is geometric: gradient trajectories are understood from the manifold.

- Generating families of Legendrian submanifolds are not really tractable.

Differential of the generating family chain complex is not understood from the Legendrian submanifold.

2. Henry-Rutherford limiting process

Goal. Deforming the Riemannian metric to constrain the difference function gradient flow on the Legendrian submanifold.

- Pick a Riemannian metric $g = g_B \oplus g_F$ on $B \times \mathbb{R}^{2N}$.
- For $s \in (0,1]$, let us define $g_s = (s^{-1} g_B) \oplus g_F$.
- Take the adiabatic limit $s \to 0$ in the g_s-gradient flow of S.

Analysis of the slow-fast systems.

- $$\begin{cases} \partial_t b_s(t) = -s \nabla_{g_B} S(b_s(t), \eta_s(t)) \\ \partial_t \eta_s(t) = -\nabla_{g_F} S(b_s(t), \eta_s(t)) \end{cases}$$

- In the slow-time $\tau = st$ scale:

$$\begin{cases} \partial_\tau b_s(\tau) = -\nabla_{g_B} S(b_s(\tau), \eta_s(\tau)) \\ s\partial_\tau \eta_s(\tau) = -\nabla_{g_F} S(b_s(\tau), \eta_s(\tau)) \end{cases}$$

As long as $s \neq 0$, the two systems are equivalent, but in the limit $s \to 0$:

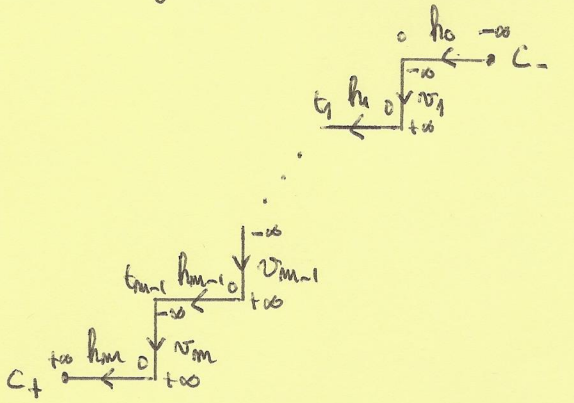
- Solutions of the fast system are drawn in a fibre (vertical fragments).
- Solutions of the slow system are drawn on $\{\nabla_{g_F} S = 0\}$ (horizontal fragments).

Two distinct states whether genuine gradient trajectories of S are close to $\{\nabla_{g_F} S = 0\}$ or not.
From now on, $\{\nabla_{g_F} S = 0\}$ is identified with $\Lambda \times_B \Lambda$.

Definition. (F.)
A gradient staircase from c_- to c_+ is a tuple $e = (h_0, v_1, h_1, \dots, v_m, h_m)$ where

- h_i are horizontal fragments;
- v_j are vertical fragments;
- concatenation of the fragments of e is continuous;
- h_0 starts at c_- and h_m ends at c_+.

The set of all gradient staircases from c_- to c_+ is $\mathcal{M}^{st}(c_-, c_+)$.



- Horizontal fragments are uniquely determined by the front projection of Λ.
- Vertical fragments are understood from fibrewise gradient trajectories of the generating families (birth/death of critical points, handleslides, etc.).

Definition. (F.)

Let $\gamma_k \in \mathcal{M}(c_-, c_+; g_{s_k})$ with $s_k \to 0$ and $e = (h_0, v_1, h_1, \dots, v_m, h_m) \in \mathcal{M}^{st}(c_-, c_+)$.

Then, $(\gamma_k)_{k \in \mathbb{N}}$ Floer-Gromov converges towards e whenever the following two conditions are met:

1. There exist $(\tau_k^{v_1})_{k \in \mathbb{N}}, \dots, (\tau_k^{v_m})_{k \in \mathbb{N}}$ such that $\gamma_k(\cdot + \tau_k^{v_i}) \xrightarrow[k \to +\infty]{} v_i$ in the C^1_{loc}-topology.
2. There exist $(\tau_k^{h_0})_{k \in \mathbb{N}}, \dots, (\tau_k^{h_m})_{k \in \mathbb{N}}$ such that $\gamma_k(s_k^{-1}(\cdot + \tau_k^{h_j})) \xrightarrow[k \to +\infty]{} h_j$ in the C^1_{loc}-topology.

Floer-Gromov convergence towards gradient staircases chains is defined similarly.

Remark.

- Time-shifts are here to take into account that $\mathbb{R}^N$ acts freely by time-translations on the difference function gradient flow.
- Scaling allows to recover non-constant horizontal fragments.

II. Sketch of the proof of the compactness theorem

1. An infinite bubbling-like phenomenon

General case

When a non-compact group acts freely on a partial differential equation, the moduli spaces of its solutions are compactified by adding broken objects obtained by concatenating solutions. When solutions consume a predetermined amount of some kind of energy available in finite total amount, broken solutions automatically have a finite number of fragments.

Henry-Rutherford limiting process

Vertical fragments can become arbitrarily short near singularities of the front projection.
$\rightsquigarrow$ Vertical fragments do not consume a predetermined amount of the finite total amount of vertical length available.
Nothing seems to prevent the Henry-Rutherford limiting process from converging to a broken gradient trajectory with infinitely many fragments.

Solution.

If the Henry-Rutherford limiting process recovers a broken gradient trajectory with infinitely many fragments, then its horizontal fragments have arbitrarily deep tangency with the singular locus of the front projection.
But this phenomenon is not generic.
A vector field is at most tangent at order n with a n-dimensional submanifold.
$\rightsquigarrow$ Definition of gradient generic Legendrian submanifolds.

Assume that Λ is front generic and that $\pi_B : \Lambda \to B$ has only Whitney pleat singularities.
$\rightsquigarrow$ There exists a codimension 1 submanifold Λ^c of Λ such that:

- $\pi_B : \Lambda \setminus \Lambda^c \to B$ is a local diffeomorphism;
- $\pi_B : \Lambda^c \to B$ is a self-transverse immersion.

Let $S_1, \dots, S_n$ be the connected components of $\Lambda \setminus \Lambda^c$.
There exist unique functions $f_1 : \overline{S_1} \to \mathbb{R}, \dots, f_n : \overline{S_n} \to \mathbb{R}$ such that S_i is locally the graph of $j^1(f_i \circ \pi_B^{-1})$.

Vertical height gradient flow in B.

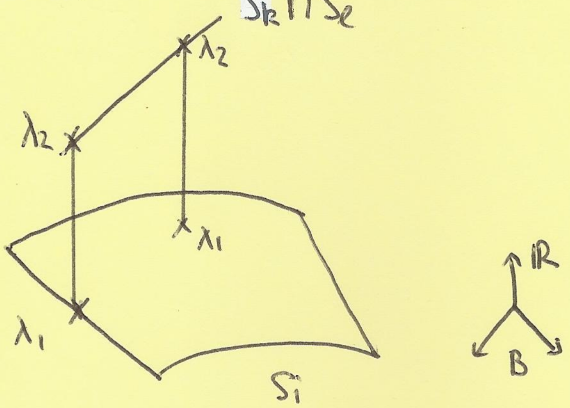
$$X_{i,k}(\lambda_1, \lambda_2) = T_{\lambda_1}\pi_B(\nabla f_i(\lambda_1)) - T_{\lambda_2}\pi_B(\nabla f_k(\lambda_2)) \in T_{\pi_B(\lambda_1)}B = T_{\pi_B(\lambda_2)}B, \; (\lambda_1, \lambda_2) \in \overline{S_i} \times_B \overline{S_k}.$$

Iterated tangency loci of $X_{i,k}$ with $\Pi_B(\Lambda^<)$

$\Pi_B^{[2]}(\lambda_1,\lambda_2)=\Pi_B(\lambda_2)$, $\beta=\{k,\ell\}$.

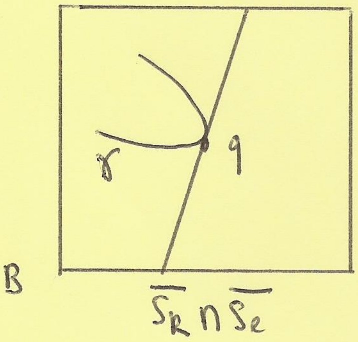
Define $L_m^{i,\beta}$ recursively on m:

- $m=0$: $L_0^{i,\beta}=\overline{S_i}\times_B(\overline{S_k}\cap\overline{S_\ell})$



- $m\geq 0$: Assume that $L_m^{i,\beta}$ is a manifold (with boundary).

$$L_{m+1}^{i,\beta}=\{(\lambda_1,\lambda_2)\in L_m^{i,\beta}\ ;\ X_{i,k}(\lambda_1,\lambda_2)\in T_{(\lambda_1,\lambda_2)}\Pi_B^{[2]}(T_{(\lambda_1,\lambda_2)}L_m^{i,\beta})\}.$$



$\Pi_B(\lambda_1)=q=\Pi_B(\lambda_2)$

$\dot\gamma=X_{i,k}\circ\gamma$

<u>Definition.</u> (F.)

Λ is gradient generic whenever for all m, i, β, $L_m^{i,\beta}$ is a well-defined manifold (with boundary) of dimension $n-m-1$.

<u>Theorem.</u> (F.)

The subset of gradient generic Legendrian submanifolds is open and dense in the set of all Legendrian submanifolds with only Whitney pleat singularities for the C^∞-topology.

2. Sketch of the proof of the finiteness of vertical fragments

<u>Theorem.</u> (F.)

Let $\gamma_k\in\mathcal{M}(c_-,c_+;g_{s_k})$ with $s_k\to 0$.

If Λ is gradient and chord generic, then only a finite number of pairwise non-equivalent vertical fragments can be recovered from $(\gamma_k)_{k\in\mathbb{N}}$.

Sketch of a proof.

Assume, for the sake of contradiction, that there exists a sequence $(v_j)_{j\in\mathbb{N}}$ of pairwise non-equivalent vertical fragments recovered from $(\gamma_k)_{k\in\mathbb{N}}$.

- Step 1. There exist a subsequence of $(v_j)_{j\in\mathbb{N}}$ and $\sigma\in(\Lambda^<\times_B\Lambda)\cup(\Lambda\times_B\Lambda^<)$ such that $v_j \xrightarrow[j\to+\infty]{} \sigma$ in the C^0-topology.

Indeed, for energetical reasons the series of the vertical lengths of $(v_j)_{j\in\mathbb{N}}$ is summable.

- Step 2. There exists a sequence of horizontal fragments $(h_j)_{j\in\mathbb{N}}$ such that
  1. h_j is parametrized by $[0,t_j]$;
  2. h_j is recovered from $(\gamma_k)_{k\in\mathbb{N}}$;
  3. there exists $t_j^*\in[0,t_j]$ such that $h_j(t_j^*)\in(\Lambda^<\times_B\Lambda)\cup(\Lambda\times_B\Lambda^<)$.

  Moreover, $t_j \xrightarrow[j\to+\infty]{} 0$ and $h_j \xrightarrow[j\to+\infty]{} \sigma$ in the C^0-topology.

Sketch of a proof.

- Step 1. By chord genericity, there exists an open neighbourhood U of σ such that $\mathrm{bit}\,\delta\cap U=\emptyset$. Since $v_j \xrightarrow[j\to+\infty]{} \sigma$, there exists a subsequence such that v_j has range in $\mathcal{U}$.
- Step 2. Recover horizontal fragments h_j such that $h_j(0)=v_j^+$.
- Step 3. Recover vertical fragments w_j such that $h_j(t_j)=w_j^-$. For energetical reasons, there exists a subsequence such that w_j has range in $\mathcal{U}$.
- Step 4. Since Λ has only Whitney pleat singularities, shrink U such that $\Pi_B(U)\setminus\Pi_B(U\cap((\Lambda^<\times_B\Lambda)\cup(\Lambda\times_B\Lambda^<)))$ has exactly two connected components. For connexity reasons, there exists $t_j^*\in[0,t_j]$ such that $h_j(t_j^*)\in(\Lambda^<\times_B\Lambda)\cup(\Lambda\times_B\Lambda^<)$.
- Step 5. For energetical reasons, $t_j \xrightarrow[j\to+\infty]{} 0$.
- Step 6. Since $v_j \xrightarrow[j\to+\infty]{} \sigma$ in the C^0-topology and $t_j \xrightarrow[j\to+\infty]{} 0$, $h_j \xrightarrow[j\to 0]{} \sigma$ in the C^0-topology. $\blacksquare$

- Step 3. The sequence of horizontal fragments $(h_j)_{j\in\mathbb{N}}$ contradicts gradient genericity.

Sketch of a proof.

Perhaps exchanging the generating families in δ, there exist i and β such that $\sigma\in L_0^{i,\beta}$.

Step 1. For all $\theta>0$, let us define $L_m^{i,\beta}(\theta)$ by induction:

- $m=0$: $L_0^{i,\beta}(\theta)=L_0^{i,\beta}$;
- $m\geqslant 0$: $L_{m+1}^{i,\beta}(\theta)$ is the set of points at which the angle between $X_{i,\beta}$ and $T\Pi_B^{[2]}(L_m^{i,\beta})$ is $\leqslant\theta$.

By construction, $L_m^{i,\beta}(\theta)$ is an open neighbourhood of $L_m^{i,\beta}$.
In particular, $\sigma \in L_0^{i,\beta}(\theta)$.

. Step 2. For all m, there exists a subsequence such that

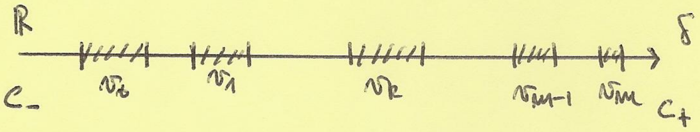
$$h_j(t_j^*),\ h_{j+1}(t_{j+1}^*) \in L_m^{i,\beta}(\theta) \Rightarrow h_j(t_j^*) \in L_{m+1}^{i,\beta}(\theta).$$

Indeed, if $h_j(t_j^*) \notin L_{m+1}^{i,\beta}(\theta)$ and t_j is small enough, then $h_j(t_j) \notin L_m^{i,\beta}(\theta)$.
Intuitively picture a flow line leaving a submanifold with a certain angle, then the time taken by the flow line to come back to the submanifold is at least proportionnal to this angle.
Moreover, if $h_j(t_j)$ and $h_{j+1}(t_{j+1}^*)$ are close enough (recall that $h_j \xrightarrow[j\to+\infty]{} \sigma$); then $h_{j+1}(t_{j+1}^*) \notin L_m^{i,\beta}$, which is a contradiction.

. Step 3. For all m, there exists J_m such that $\forall j \geq J_m$, $h_j(t_j^*) \in L_m^{i,\beta}(\theta)$.
If $\theta > 0$ is small enough, then $L_n^{i,\beta}(\theta)$ is empty, since $L_n^{i,\beta}$ is.
But since $h_j \xrightarrow[j\to+\infty]{} \sigma$, applying Step 3 shows that $\sigma \in L_n^{i,\beta}(\theta)$, a contradiction. ■ ■

3. Sketch of the proof of the compactness theorem.

.Step 1. By finiteness, recover all the non-equivalent vertical fragments $(v_1, \dots, v_m)$ ordered by decreasing values of δ.



.Step 2. In the complement of the union of the intervals provided by Step 1, recover all horizontal fragments $(h_0, \dots, h_m)$.
In particular, by exhaustivity of the vertical fragments provided by Step 1:

$$\forall k \in \{1, \dots, m-1\},\ h_k^- = v_k^+,\ h_k^+ = v_{k+1}^-,$$

and $h_0^- = c_-$, $h_0^+ = v_1^+$, $h_m^- = v_m^+$ and $h_m^+ = c_+$.

.Step 3. Construct a gradient staircases chain from the data provided by Steps 1 and 2.
The subset of k such that either h_k^- or h_k^+ is a critical point of δ provides a partition

$$\begin{cases} e_1 = (h_{0,1}, v_{1,1}, h_{1,1}, \dots, v_{m_1,1}, h_{m_1,1}), \\ \quad \dots \\ e_k = (h_{0,k}, v_{1,k}, h_{1,k}, \dots, v_{m_k,k}, h_{m_k,k}), \\ \quad \dots \\ e_r = (h_{0,r}, v_{1,r}, h_{1,r}, \dots, v_{m_r,r}, h_{m_r,r}), \end{cases}$$

of the ordered tuple $(h_0, v_1, h_1, \dots, v_m, h_m)$ with possibly missing $h_{0,k}$ or $h_{m_k,k}$.
If so, then $h_{0,k}$ or $h_{m_k,k}$ is defined to be a constant horizontal fragment defined on a

semi-infinite interval and equal to the relevant critical point of δ.

• Step 4. Check that the endpoints of successive fragments match.
Chord genericity of Λ ensures that the set of critical values of δ is finite and injective. ∎

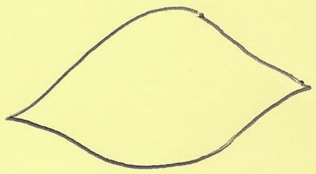
III. Homology computations with gradient staircases

1. Construction of generating families

Standard Legendrian unknotted sphere

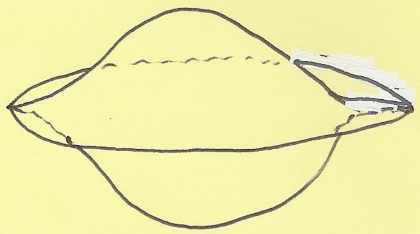
Let us define Λ_0^n recursively on n:

- $n=1$: Λ_0^1 is the maximal Thurston-Bennequin number Legendrian unknot.



- $n \geq 1$: Λ_0^{n+1} is defined by its front projection.

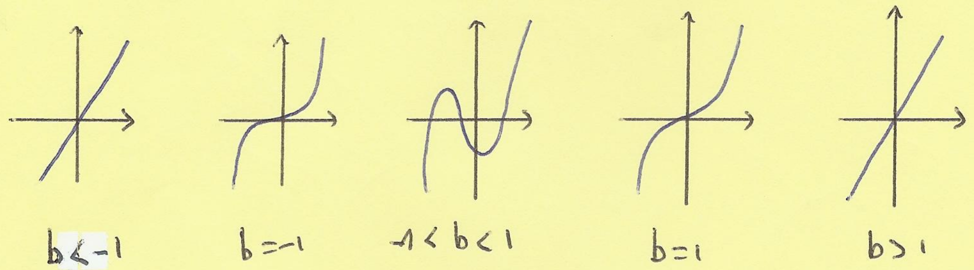
The front projection of Λ_0^{n+1} is obtained by spinning the front projection of Λ_0^n around the z-axis.



$f_0^n(b,\eta) = \eta^3 - 3(\|b\|^2 - 1)\eta$ is a generating family of Λ_0^n.

Using cut-off functions: $f_0^n \rightsquigarrow \tilde{f}_0^n$ generating family of Λ_0^n

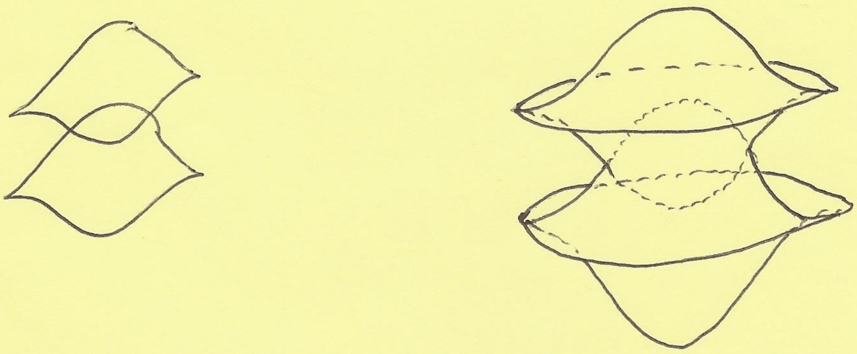
$\tilde{f}_0^n = A$ outside a compact subset, A nonzero linear form



Higher dimensional Legendrian Hopf links

Let us define $\Lambda_{(2)}^n$ by its front projection.

Its front projection is obtained by taking two overlapping vertical copies of the front projection of Λ_1^n.

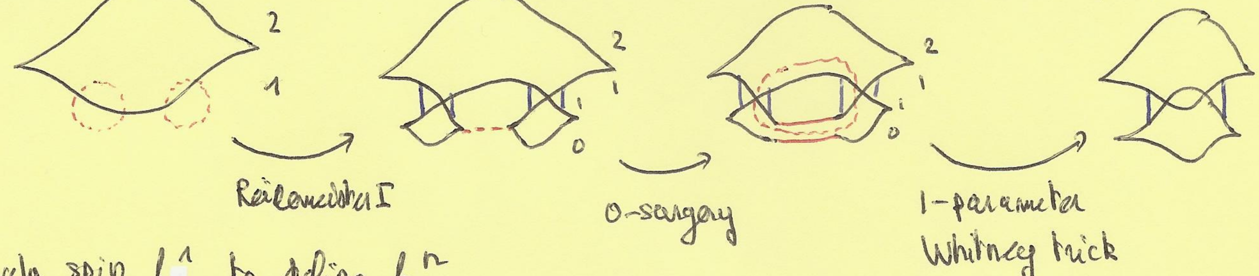


Generating family $f_{//}^n$: parallel copy construction

- Step 1. Shrink $\tilde{f}_0^1$ to define a linear-at-infinity generating family f_1^1 of $\frac{1}{2}\Lambda_0^1$.
- Step 2. Stabilise $\tilde{f}_0^1$ to produce a +1 shift of its fibrewise Morse indices and use a fibre-preserving diffeomorphism to recover a linear-at-infinity generating family f_0^2 of Λ_0^1.
- Step 3. Stabilise f_1^1 and use a fibre-preserving diffeomorphism to recover a linear-at-infinity generating family $\tilde{f}_1^1$ of Λ_0^1 such that:
  - the fibrewise Morse indices of $\tilde{f}_0^1$ and $\tilde{f}_1^1$ are equal;
  - the fibre dimensions of $\tilde{f}_1^1$ and $\tilde{f}_2^1$ are equal.
- Step 4. Shift slightly downward the values of $\tilde{f}_1^1$ to define F_1^1.
- Step 5. Translate f_2^1 in the fibre to define F_2^1 such that $\operatorname{supp} F_1^1 \cap \operatorname{supp} F_2^1 = \emptyset$.
- Step 6. Define $f_{//}^1 = \begin{cases} F_1^1, & \text{on } \operatorname{supp} F_1^1, \\ F_2^1, & \text{on } \operatorname{supp} F_2^1, \\ A, & \text{elsewhere.} \end{cases}$
- Step 7. Iteratively spin $f_{//}^1$ to define $f_{//}^n = F_1^n + F_2^n$.

Generating family $f_\#^n$: surgery construction

- Stabilise $\tilde{f}_0^1$ to produce a +1 shift of its fibrewise Morse indices and use a fibre-preserving diffeomorphism to recover a linear-at-infinity generating family f_1^1 of Λ_0^1.
- Follow the following steps to define a linear-at-infinity generating family $f_\#^1$ of $\Lambda_{(2)}^1$.



- Iteratively spin $f_\#^1$ to define $f_\#^n$.

2. Homological computations.

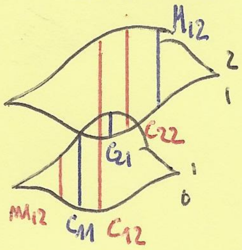
Generators and grading of the chain complex.

- Generators: Reeb chords of $\Lambda_{(2)}^n$.

One Reeb chord by pair of sheets of $\Lambda_{(2)}^n$: $\binom{4}{2} = 6$ generators.

Perturb $\Lambda_{(2)}^n$ to make it chord generic: offset the Reeb chords out of the central axis.

- Grading: $\mu = \Delta$ Fibrewise Morse indices + Morse index height $- 1$



Critical point	c_{12}	c_{11}	c_{22}	M_{12}	c_{21}	m_{12}
Grading	$n+1$	n	n	n	$n-1$	0

If $n \geq 2$, using grading:
- $\partial c_{12} \in \langle c_{11}, c_{22}, M_{12} \rangle$;
- $\partial c_{11}, \partial c_{22}, \partial M_{12} \in \langle c_{21} \rangle$;
- $\partial m_{12} = \partial c_{21} = 0$.

Identifying gradient staircases.

- Gradient staircases decrease chord length.
- Gradient staircases are uniquely determined by their vertical fragments.

$\rightsquigarrow$ It suffices to examine the strands between which the generators of the chain complex lie.

- Vertical fragments are made from:
  - death/birth flow lines;
  - handleslides;
  - concatenations of the two.

Simple generating family homology of $b_{//}^n$.

Proposition. (F.)

If $n \geq 2$: $\begin{cases} \partial_{//}^n c_{12} = M_{12} \\ \partial_{//}^n c_{11} = \partial_{//}^n c_{22} = \partial_{//}^n M_{12} = \partial_{//}^n c_{21} = \partial_{//}^n m_{12} = 0 \end{cases}$

Therefore: $GFH_\cdot(b_{//}^n) = \langle c_{11}, c_{22}, c_{21}, m_{12} \rangle = \mathbb{F}_2^2[n] \oplus \mathbb{F}_2[n-1] \oplus \mathbb{F}_2[0]$.

Proof.

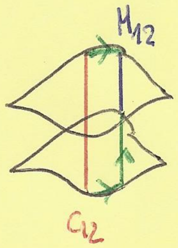
No handleslides.

$\rightsquigarrow$ Vertical fragments do not jump between strands of the different connected components of $\Lambda_{(2)}^n$.

$\rightsquigarrow$ $\partial_{//}^n$ must preserve the indices decorating the generators.

Only possible gradient staircases are between c_{12} and M_{12}.

There is a unique gradient staircase between c_{12} and M_{12} :



$\blacksquare$

Proposition. (F.)

If $n \geq 2$: $\begin{cases} \partial_{\#}^n c_{12} = c_{11} + c_{22} + M_{12} \\ \partial_{\#}^n c_{11} = \partial_{\#}^n c_{22} = c_{21} \\ \partial_{\#}^n M_{12} = \partial_{\#}^n c_{21} = \partial_{\#}^n m_{12} = 0 \end{cases}$

Therefore: $GFH_\cdot(b_{\#}^n) = \langle c_{11} + c_{22}, m_{12} \rangle = \mathbb{F}_2[n] \oplus \mathbb{F}_2[0]$.

Proof.

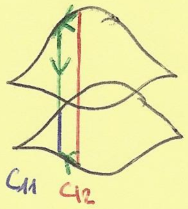
Vertical fragments can now jump between strands of different connected components of $\Lambda_{(2)}^n$.

$\rightsquigarrow$ $\partial_{\#}^n$ no longer preserves the indices decorating the generators.

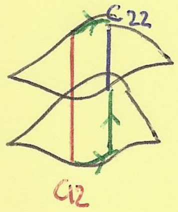
All the previous gradient staircases remain.

New gradient staircases.

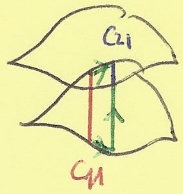
- There is a unique gradient staircase between c_{12} and c_{11}:



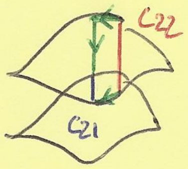
- There is a unique gradient staircase between c_{12} and c_{22}:



- There is a unique gradient staircase between c_{11} and c_{21}:



- There is a unique gradient staircase between c_{22} and c_{21}:



■

Mixed generating family homology of $(b_{11}^n, b_{\#}^n)$.

Proposition. (F.)

If $n \geq 2$,
$$\begin{cases} \partial^n_{11,\#} c_{12} = c_{22} + M_{12} \\ \partial^n_{11,\#} c_{11} = c_{21} \\ \partial^n_{11,\#} c_{22} = \partial^n_{11,\#} M_{12} = \partial^n_{11,\#} c_{21} = \partial^n_{11,\#} m_{12} = 0 \end{cases}$$

Therefore: $GFH.(b_{11}^n, b_{\#}^n) = \langle M_{12}, m_{12} \rangle = \mathbb{F}_2[n] \oplus \mathbb{F}_2[0]$.

Proof.

b_{11}^n has no handleslides and $b_{\#}^n$ appears with a minus sign in the difference function.
$\rightsquigarrow$ Remove all the previous gradient staircases with a downward handleslide of $b_{\#}^n$. ■

b_{11}^n and $b_{\#}^n$ give rise to linear-at-infinity generating families F_{11}^n and $F_{\#}^n$ of higher-dimensional right-handed Legendrian trefoil knot such that:
$$\begin{cases} GFH.(F_{11}^n) = \mathbb{F}_2[n] \oplus \mathbb{F}_2[n-1] \oplus \mathbb{F}_2[0] = GFH.(F_{\#}^n). \\ GFH.(F_{11}^n, F_{\#}^n) = \mathbb{F}_2[0] \end{cases} \quad \text{(F.)}$$

This discussion is the main starting point to understand the geography of mGFH.

Research prospects

1. Gluing conjecture

The correspondence between $\mathcal{M}^{st}(c_-, c_+)$ and $\mathcal{M}(c_-, c_+; g_s)$ now amounts to a gluing conjecture.

Conjecture. (F.)

If Λ is generic and $e \in \mathcal{M}^{st}(c_-, c_+)$, then there exists a one-parameter family $\gamma_s \in \mathcal{M}(c_-, c_+; g_s)$ with $s \to 0$ such that $\gamma_s \xrightarrow[s \to 0]{} e$ in the Floer-Gromov topology.

Moreover, if $\dim \mathcal{M}^{st}(c_-, c_+) = 0$, then the above one-parameter family is unique.

Strategy of the proof.

- Step 1. Floer-Gromov convergence is a convergence in some weighted Sobolev spaces. The proof of the compactness theorem provides quantitative asymptotic estimates for the Henry-Rutherford limiting process that almost lead this step.
- Step 2. Construct a right-invertible Fredholm operator whose solutions are gradient staircases.
- Step 3. Construct a smooth approximation of e (pre-gluing).
- Step 4. Apply the Newton-Raphson method to the Fredholm operator and the pre-gluing.

2. mGFH is a complete invariant for generating families

Observe that mGFH fits in the following long exact sequence:

$$\dots \to GFH_k(f_1, f_2) \xrightarrow{\tau_k} H_k(\Lambda; \mathbb{F}_2) \xrightarrow{\sigma_k} GFH^{n-k}(f_2, f_1) \xrightarrow{\rho_k} \dots,$$

induced by the short exact sequence of the triple $(\delta^{w}, \delta^{\varepsilon}; \delta^{-\varepsilon})$ (Bourgeois-Sabloff-Traynor, 2015, F.).

If Λ is connected and $f_1 \sim f_2$, then τ_n is surjective (Bourgeois-Sabloff-Traynor, 2015, F.). The converse is at least true on some relevant examples (F.) and it is conjectured to be true in all generality.

Conjecture. (F.)

If Λ is connected, then τ_n is surjective if, and only if, $f_1 \sim f_2$.

3. Geography questions for GFH and mGFH

The graded $\mathbb{F}_2$-vector space geography of GFH is already known (Bourgeois-Sabloff-Traynor, 2015).

Question. (F.)

What are the graded $\mathbb{F}_2$-vector spaces that can be realised by mGFH?

There exists a ring structure on GFH^{*} (Myer, 2018)

Question. (F.)

What are the possible ring structures on GFH^{*}?