Generating family homologies of Legendrian submanifolds and moduli spaces of gradient staircases

Doctoral thesis defence

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Some context and motivation

Contact manifolds and Legendrian submanifolds

A *contact manifold* is a pair (M^{2n+1}, ξ) where

- M is a smooth manifold of odd dimension $2n+1$; and
- ξ is locally $\ker(\alpha)$ with $\alpha \in \Omega^1(M)$ such that $\alpha \wedge (d\alpha)^n \neq 0$;

It is said that ξ is a *contact structure* and α is a *contact form*.

Contact structures are as far away as possible from foliations.

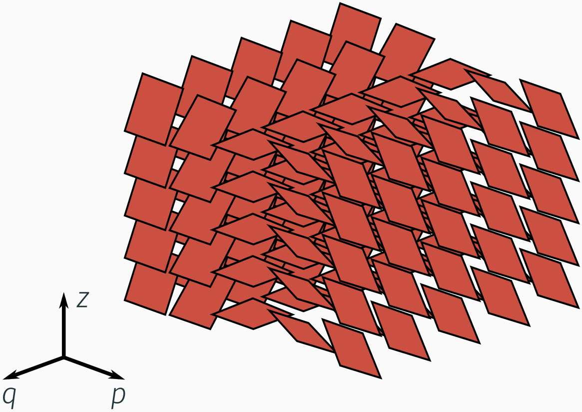
Proposition

If a submanifold $L \subset M$ is such that $TL \subset \xi$, then $\dim(L) \leqslant n$.

Such a submanifold Λ is *Legendrian* when $\dim(\Lambda)$ is maximal.

Tautological contact topology

$\xi_B = \ker(\mathrm{d}\,z - p\,\mathrm{d}\,q)$ is a contact structure on $J^1B = T^*B_{(q,p)} \times \mathbf{R}_z$.



Contact manifolds (M^{2n+1}, ξ) are locally modelled on $(J^1\mathbf{R}^n, \xi_{\mathbf{R}^n})$.

Drawing Legendrian submanifolds

The *front projection* of Λ is its image by $J^1 B \to J^0 B = B \times \mathbf{R}$.

Front projections of Legendrian submanifolds are often singular, even though Legendrian submanifolds are smooth.

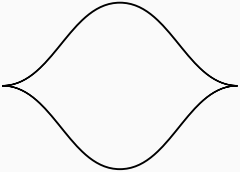
Proposition

It is possible to perturb Λ in a way that its front projection is stratified by transversally intersecting submanifolds, so that:

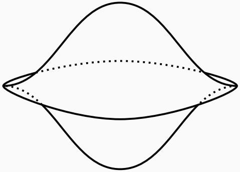
$$p_i = \frac{\partial z}{\partial q_i},$$

and in particular, Λ is uniquely determined by its front.

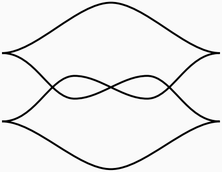
Some important front projections



$\Lambda_0^1 \subset (J^1\mathbf{R}, \xi_{\mathbf{R}})$



$\Lambda_0^2 \subset (J^1\mathbf{R}^2, \xi_{\mathbf{R}^2})$



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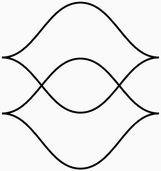
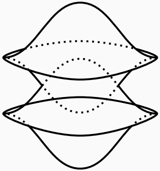


$\Lambda_{(2)}^1 \subset (J^1\mathbf{R}, \xi_{\mathbf{R}})$



$\Lambda_{(2)}^2 \subset (J^1\mathbf{R}^2, \xi_{\mathbf{R}^2})$

Contact rigidity of Legendrian submanifolds

Legendrian isotopies are not reducible to homotopy theory.

Theorem

Smooth isotopy classes of Legendrian submanifolds split into infinitely many distinct Legendrian isotopy classes.

How to study the contact rigidity of Legendrian submanifolds?

Three types of different techniques:

- Pseudo-holomorphic curves theory;
- Generating families; or
- Microlocal sheaf theory;

and they are conjectured to produce ”equivalent” invariants, but invariants from GF currently lack of algebraic structure.

Moduli spaces in generating family theory

How to construct invariants from generating families?

- From a generating family, construct a function $\delta\colon B \times F \to \mathbf{R}$.
- Pick a product Riemannian metric $g = g_B \oplus g_F$ on $B \times F$.
- For all critical points c_- and c_+ of δ, study:

$$\mathcal{M}(c_-, c_+; g) = \left\{ \gamma; \dot{\gamma} = -\nabla_g \delta \circ \gamma \text{ and } \lim_{\pm\infty} \gamma = c_\pm \right\},$$

which are the *Morse moduli spaces* associated to δ and g.

Henry–Rutherford combinatorial dg-algebra

There exists a dg-algebra for Legendrian knots constructed from moduli spaces of chord paths $\mathcal{M}^{\mathcal{C}}(c_-, c_+)$ and it has a potential geometrical grounding in GF **[Henry–Rutherford, 2013]**.

So-called moduli spaces of gradient staircases $\mathcal{M}^{\mathrm{st}}(c_-, c_+)$ are expected to make a bridge between $M^{C}(c_-, c_+)$ and $M(c_-, c_+; g)$.

Conjecture (Correspondence, Conjecture 3.1, p. 63, H.-R., '13)

If $s \in (0, 1]$ is small enough and $g_s = (s^{-1} g_{\mathbf{R}}) \oplus g_F$, then there exists a one-to-one set correspondence between $\mathcal{M}^{\mathrm{st}}(c_-, c_+)$ and $\mathcal{M}(c_-, c_+; g_s)$, provided they are both finite.

Whenever finite, $\mathcal{M}^{\mathcal{C}}(c_-, c_+) \simeq \mathcal{M}^{\mathrm{st}}(c_-, c_+)$ **[H.-R., '13]**.

Main statement

It is difficult to define chord paths $M^C(c_-, c_+)$ in any dimensions, but asking whether $\mathcal{M}^{\mathrm{st}}(c_-, c_+)$ and $\mathcal{M}(c_-, c_+)$ are in bijective correspondence or not still makes sense.

This question is adressed by developping a compactness-gluing strategy for the adiabatic limit $s \to 0$.

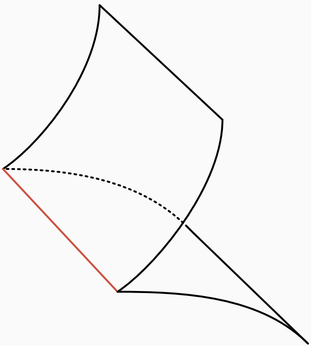
Theorem (Compactness, Theorem 4.1, p. 69, F.)

If $\Lambda \to B$ has only Whitney pleat singularities and Λ is generic, then for $\gamma_s \in \mathcal{M}(c_-, c_+; g_s)$, with $g_s = (s^{-1} g_B) \oplus g_F$ and $s \to 0$, there exist $s_k \xrightarrow[k \to +\infty]{} 0$ and $e \in \overline{\mathcal{M}^{\mathrm{st}}}(c_-, c_+)$ such that

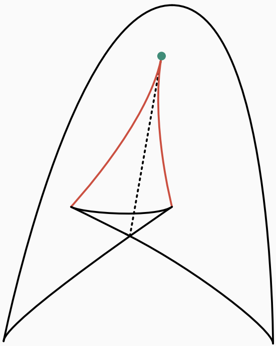
$$\gamma_{s_k} \xrightarrow[k \to +\infty]{} e,$$

in the Floer–Gromov topology.

Main statement



Whitney pleat



Swallowtail

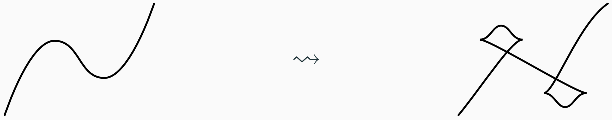
- When $n = 1$, the singularity assumption is empty.
- When $n \geqslant 2$, it is homotopical [Alvarez Gavela, 2016].

Henry–Rutherford limiting process and gradient staircases

Generating families

Holonomic sections of (J^1B, ξ_B) are Legendrian isotopic to 0_B.

If $\Lambda \sim 0_B$, then Λ is not necessarily a holonomic section of J^1B.



Legendrian isotopy

Λ is a graph reduction of some holonomic section of $J^1(B \times \mathbf{R}^N)$.

Definition

A *generating family* of Λ is a map $f\colon B_b \times \mathbf{R}^N{}_\eta \to \mathbf{R}$ such that

1. $\Sigma_f = (\partial_\eta f)^{-1}(\mathbf{0})$ is a transversally cut-out submanifold; and
2. $\Sigma_f \ni (b, \eta) \mapsto (b, \partial_b f(b,\eta), f(b,\eta)) \in \Lambda$ is a diffeomorphism.

Homotopy lifting property of generating families

If Λ has a generating family, then it has infinitely many others:

- **Stabilisation**. Change $\dim(F)$ and fibrewise Morse indices μ_F.
- **Fibred diffeomorphism**. Deform fibrewise critical sets Σ_f.

Generating families are *equivalent* whenever they can be made equal by applying a finite number of these moves.

Generating families are relevant to study Legendrian isotopies.

Theorem (Chaperon, Chekanov, Laudenbach, Sikorav, '84–'96)

Isotopic Legendrian submanifolds have equivalent GF.

Difference function of generating families

Difference function. $\delta(b, \eta_1, \eta_2) = f_1(b, \eta_1) - f_2(b, \eta_2)$.

Proposition

The critical points of δ are of two types:

1. *The positively/negatively valued critical points of δ are in one-to-one correspondence with the Reeb chords of Λ.*
2. *There exists $\varepsilon > 0$ such that δ is Morse–Bott in $\{-\varepsilon < \delta < \varepsilon\}$, and it has a unique critical submanifold Σ in this region. Moreover, Σ is diffeomorphic to Λ.*

Idea.

Construct invariants of Λ from the Morse theory of δ.

Generating family homologies: GFH and mGFH

Morse theory is ill behaved on noncompact manifolds.

$\rightsquigarrow$ Need to tame the behaviour at infinity of GF.

$\rightsquigarrow$ **Assumption.** Generating families are linear-at-infinity.

Definition (Traynor, 2001)

The *generating family homology* of (f_1, f_2) is defined by

$$\mathrm{GFH}_\bullet(f_1, f_2) = H_{k+N+1}(\{\delta < \omega\}, \{\delta < \varepsilon\}; \mathbf{F}_2),$$

where all the positive critical values of δ are located in (ε, ω).

- **Generators.** Positively valued critical points of δ.
- **Grading.** Morse index of δ shifted by $-N-1$.
- **Differential.** On generators: $\partial c_- = \sum_{c_+} \#_{\mathbf{F}_2} \mathcal{M}(c_-, c_+) c_+$.

Invariance of GFH and mGFH

Grading shift ensures the invariance of GFH.

Theorem (Traynor–Sabloff, 2013)

If $f_1 \sim f_2$, then $\mathrm{GFH}_\bullet(f_1) \simeq \mathrm{GFH}_\bullet(f_2)$ as graded vector spaces.

Assymmetry breaks the grading invariance of mGFH.

Theorem (Invariance of mGFH, Theorem 2.2, p. 27, F.)

If $f_1 \sim f_2$, then there exist $N_1, N_2 \in \mathbb{Z}$ such that for all f:

$$\mathrm{GFH}_\bullet(f_1, f) \simeq \mathrm{GFH}_{\bullet+N_1}(f_2, f),$$
$$\mathrm{GFH}_\bullet(f, f_1) \simeq \mathrm{GFH}_{\bullet+N_2}(f, f_2),$$

as graded vector spaces.

Henry–Rutherford limiting process

Generating family homology is hard to compute:

- Generating families are often qualitatively described.
- Difference function gradient flows are hardly tractable.
- Differential of the GF chain complex is not geometric.

Idea.

Constrain difference function gradient flow lines on Λ.

Speed up fibre components/slow down base components:

- Pick a Riemannian metric $g = g_B \oplus g_F$ on $B \times \mathbb{R}^{2N}$.
- For all $s \in (0, 1]$, let us define $g_s = (s^{-1} g_B) \oplus g_F$.
- Take the adiabatic limit $s \to 0$.

Slow-fast system associated to the adiabatic limit $s \to 0$

The adiabatic limit $s \to 0$ has a slow-fast dynamic.

Fast system

Time variable: t

$$\begin{cases} \partial_t b_s(t) = -s\nabla_{g_B}\delta(b_s(t), \eta_s(t)), \\ \partial_t \eta_s(t) = -\nabla_{g_F}\delta(b_s(t), \eta_s(t)). \end{cases}$$

$\rightsquigarrow$ $s = 0$ sol. $\subset$ fibres.

$\rightsquigarrow$ *Vertical fragments*.

[Proposition 4.1, p. 70, F.]

Slow system

Time variable: $\tau = st$

$$\begin{cases} \partial_\tau b_s(\tau) = -\nabla_{g_B}\delta(b_s(\tau), \eta_s(\tau)), \\ s\partial_\tau \eta_s(\tau) = -\nabla_{g_F}\delta(b_s(\tau), \eta_s(\tau)). \end{cases}$$

$\rightsquigarrow$ $s = 0$ sol. $\subset \{\nabla_{g_F}\delta = \mathbf{0}\}$.

$\rightsquigarrow$ *Horizontal fragment*.

[Theorem 4.2, p. 89, F.]

As long as $s \neq 0$, the slow and the fast systems are equivalent.

Observe that $\{\nabla_{g_F}\delta = \mathbf{0}\}$ discriminates the two $s = 0$ states.

Moduli spaces of gradient staircases

Definition (Definition 3.2, p. 36, F.)

A *gradient staircase* is a tuple $\mathbf{e} = (h_0, v_1, h_1, \ldots, v_m, h_m)$:

- h_i are horizontal fragments and v_j are vertical fragments;
- concatenation of fragments of $\mathbf{e}$ are continuous; and
- h_0 starts at c_- and h_m ends at c_+.

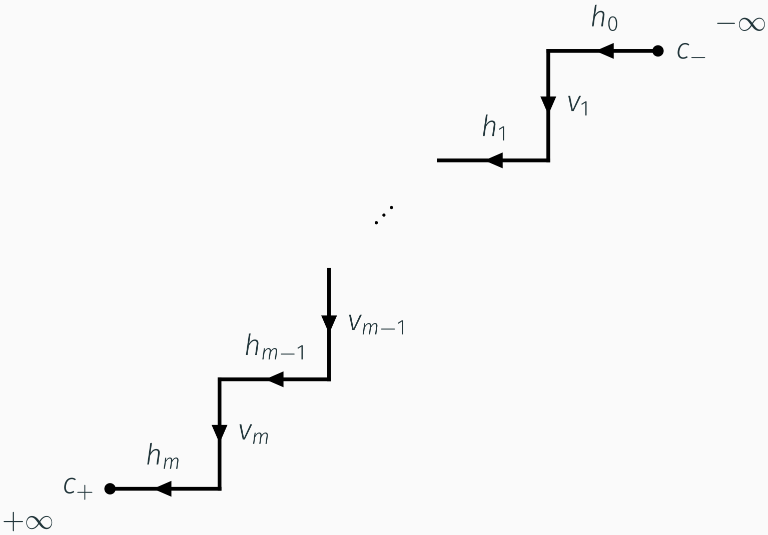
The set of gradient staircases from c_- and c_+ is $\mathcal{M}^{\mathrm{st}}(c_-, c_+)$.

$\mathcal{M}^{\mathrm{st}}(c_-, c_+)$ is more tractable than $\mathcal{M}(c_-, c_+)$, because

- horizontal fragments are uniquely determined by Λ; and
- vertical fragments are related to bifurcations of GF.

$\rightsquigarrow$ The adiabatic limit $s \to 0$ splits the contributions to GFH.

Moduli spaces of gradient staircases



$\mathbf{e} \in \mathcal{M}^{\mathrm{st}}(c_-, c_+)$

Floer–Gromov topology

Definition (Definition 3.4, p. 40, F.)

Let $\gamma_k \in \mathcal{M}(c_-, c_+; g_{s_k})$ with $s_k \xrightarrow[k\to+\infty]{} 0$ and $\mathbf{e} \in \mathcal{M}^{\mathrm{st}}(c_-, c_+)$, then $(\gamma_k)_{k\in\mathbf{N}}$ *Floer–Gromov converges* towards $\mathbf{e}$ whenever:

- There exist $(\tau_k^{v_i})_{k\in\mathbf{N}}$ such that $\gamma_k(\cdot + \tau_k^{v_i}) \xrightarrow[k\to+\infty]{C^1_{\mathrm{loc}}} v_i$.
- There exist $(\tau_k^{h_j})_{k\in\mathbf{N}}$ such that $\gamma_k({s_k}^{-1}(\cdot + \tau_k^{h_j})) \xrightarrow[k\to+\infty]{C^1_{\mathrm{loc}}} h_j$.

- Time shifts account for the free action by time-translations.
- Scaling allows to recover nonconstant horizontal fragments.

Proof of the compactness theorem

An infinite bubbling-like phenomenon

Prototypic situation

Data.
Moduli spaces: $\mathcal{M}$.
Symmetry: G noncompact.

Compactification: $\overline{\mathcal{M}}$.
Add fibred products of $\mathcal{M}$.

Finiteness of fragments.
Energy: E.
Total amount of E: finite.
Lower bounds on E: YES.

Adiabatic limit $s \to 0$

Data.
Moduli spaces: $\mathcal{M}^{\mathrm{st}}$.
Symmetry: $\mathbf{R}^{\mathbf{N}}$ (time-shifts).

Compactification: $\overline{\mathcal{M}^{\mathrm{st}}}$.
Gradient staircases chains.

Finiteness of fragments.
Energy: length ℓ.
Total amount of ℓ: finite.
Lower bounds on ℓ: NO.

Gradient generic Legendrian submanifolds

Problem.
Vertical fragments become arbitrarily short near singularities.

Notations.

- $\Lambda^{\prec}$ = codimension one singular submanifold of $\Lambda \to B$;
- $L = \Lambda \times_B \Lambda$, pairs of vertically aligned points in Λ; and
- $L^{\prec} = (\Lambda^{\prec} \times_B \Lambda) \cup (\Lambda \times_B \Lambda^{\prec})$, singular points of L.

Solution.
If the adiabatic limit $s \to 0$ recover infinitely many fragments:
$\rightsquigarrow$ Vertical fragments accumulate on $L^{\prec}$.
$\rightsquigarrow$ Horizontal fragments become shorter and closer.
$\rightsquigarrow$ Horizontal fragments have arbitrarily deep tangency with $L^{\prec}$.

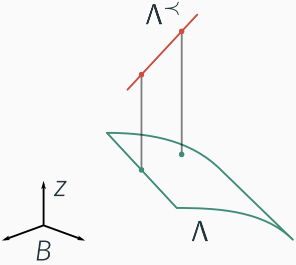
Gradient generic Legendrian submanifolds

Crucial observation.

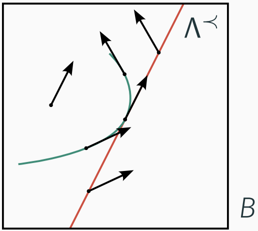
Let X be a vector field on M and let also $N \subset M$ be a submanifold. If N is generic, then X is tangent at most to order $\dim(N)$ to N.

Iterated tangency loci.

For all m, L_m is the subset of points of L at the which the height gradient vector field is tangent at least to order m to $L^{\prec}$.



$L_0 = L^{\prec}$



L_1

Gradient generic Legendrian submanifolds

Iterated tangency loci $\{L_m\}_{m \in \mathbf{N}}$ are not always well-defined.

Definition (Definition 1.11, p. 11, F.)

A Legendrian submanifold is *gradient generic* when for all m, the set L_m is well-defined and is a manifold with boundary of dimension $n - m - 1$.

Gradient genericity is a generic property.

Theorem (Transversality, Theorem 1.2, p. 12, F.)

The subset of all gradient generic Legendrian submanifolds is open and dense in the set of all Legendrian submanifolds with only Whitney pleat singularities for the C^∞-topology.

The proof relies on the appropriate use of Sard's theorem.

Finiteness of vertical fragments

Gradient genericity prevents wild breaking in the limit $s \to 0$.

Theorem (Finiteness, Theorem 4.3, p. 64, F.)

If Λ is gradient and chord generic, then only a finite number of nonequivalent vertical fragments can be recovered from any sequence $\gamma_k \in \mathcal{M}(c_-, c_+; g_{s_k})$ with $s_k \xrightarrow[k \to +\infty]{} 0$.

The proof is by contradiction, using energetical arguments.

Sketch of the proof for the finiteness of vertical fragments

$\rightsquigarrow$ Nonequivalent vertical fragments $(v_j)_{j\in\mathbf{N}}$ recovered from $(\gamma_k)_{k\in\mathbf{N}}$.

Step 1.

There exist $\sigma \in L^{\prec}$ and a subsequence such that $v_j \xrightarrow[j\to+\infty]{C^0} \sigma$.

Step 2.

There exists a sequence $(h_j)_{j\in\mathbf{N}}$ such that

- h_j is parametrised by $[0, t_j]$;
- h_j is an horizontal fragment recovered from $(\gamma_k)_{k\in\mathbf{N}}$; and
- there exists $t_j{}^* \in [0, t_j]$ such that $h_j(t_j{}^*) \in L^{\prec}$;

Moreover, $t_j \xrightarrow[j\to+\infty]{} 0$ and $h_j \xrightarrow[j\to+\infty]{C^0} \sigma$.

Sketch of the proof for the finiteness of vertical fragments

Step 3.
The sequence $(h_j)_{j\in\mathbf{N}}$ contradicts gradient genericity.

Step i.
For all $\theta > 0$, let us define $L_m(\theta)$ recursively on m by

- **Base step.** If $m = 0$, then $L_0(\theta) = L_0$.
- **Inductive step.** If $m \geqslant 0$, then $L_{m+1}(\theta)$ is the set of points at which the angle between the height gradient vector field and L_m is at most θ.

In particular, $\sigma \in L_0$ and $L_m(\theta)$ is an open neighbourhood of L_m.

Sketch of the proof for the finiteness of vertical fragments

Step ii.

For all m, there exists a subsequence such that

$$h_j({t_j}^*), h_{j+1}({t_{j+1}}^*) \in L_m(\theta) \implies h_j({t_j}^*) \in L_{m+1}(\theta).$$

Since a flow line leaving a submanifold with an angle at least θ cannot come back to it in a time less than $C\theta$, for some $C > 0$, and $t_j \xrightarrow[j\to+\infty]{} 0$ and $h_j \xrightarrow[j\to+\infty]{} \sigma$.

Step iii.

For all m, there exists J_m such that for all $j \geqslant J_m$, $h_j({t_j}^*) \in L_m(\theta)$.

Conclusion.

If $\theta > 0$ is small enough, then $L_n(\theta)$ is empty, since L_n is empty.
But since $h_j \xrightarrow[j\to+\infty]{C^0} \sigma$, then $\sigma \in L_n(\theta)$, which is a contradiction.

Sketch of the proof for the compactness theorem

Assume that Λ is gradient and chord generic.

Use that fragments are in finite number to carry the first steps.

Step 1.
Recover all vertical fragments $(v_1, \ldots, v_m)$ from $(\gamma_k)_{k \in \mathbf{N}}$.
Assume that the v_i are ordered by decreasing values of δ.

Step 2.
Recover all horizontal fragments $(h_0, \ldots, h_m)$ from $(\gamma_k)_{k \in \mathbf{N}}$.
Then, by exhaustivity of the fragments:

$$\forall k \in \{1, \ldots, m-1\}, {h_k}^- = {v_k}^-, {h_k}^+ = {v_{k+1}}^-,$$

and also ${h_0}^- = c_-$, ${h_0}^+ = {v_1}^-$, ${h_m}^- = {v_m}^+$ and ${h_m}^+ = c_+$.

Sketch of the proof for the compactness theorem

Step 3.

Construct a gradient staircases chain from the v_i and the h_j.
The set of k such that either ${h_k}^-$ or ${h_k}^+$ is a critical point of δ provides a partition:

$$\begin{cases} \mathbf{e}_1 = (h_{0,1}, v_{1,1}, h_{1,1}, \dots, v_{m_1,1}, h_{m_1,1}), \\ \dots \\ \mathbf{e}_k = (h_{0,k}, v_{1,k}, h_{1,k}, \dots, v_{m_k,k}, h_{m_k,k}), \\ \dots \\ \mathbf{e}_r = (h_{0,r}, v_{1,r}, h_{1,r}, \dots, v_{m_r,1}, h_{m_r,r}), \end{cases}$$

of $(h_0, v_1, h_1, \dots, v_m, h_m)$ with possibly missing $h_{0,k}$ or $h_{m_k,k}$.
If so, then it is defined to be constant on a semi-infinite interval.

Step 4.

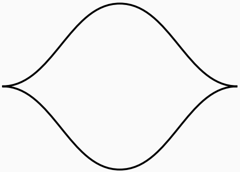
Check that the ends of successive horizontal fragments match.

Homology computations with gradient staircases

Standard Legendrian unknotted sphere

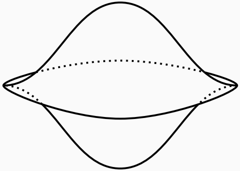
Let us define the Legendrian submanifold Λ_0^n of $(J^1\mathbf{R}^n, \xi_{\mathbf{R}^n})$ by:

Base case.

If $n = 1$, the front of Λ_0^1 is  .

Iterative step.

If $n \geqslant 1$, the front of Λ_0^{n+1} is obtained by spinning this of Λ_0^n.



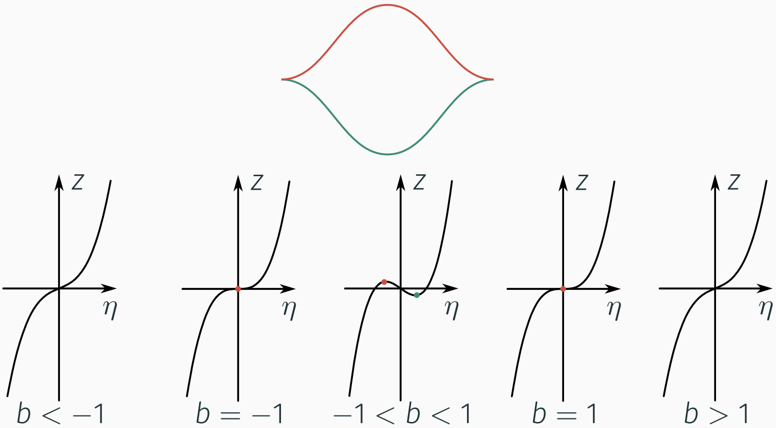
$\Lambda_0^2 \subset (J^1\mathbf{R}^2, \xi_{\mathbf{R}^2})$

A linear-at-infinity generating family of Λ_0^n

Using cut-off functions, the generating family

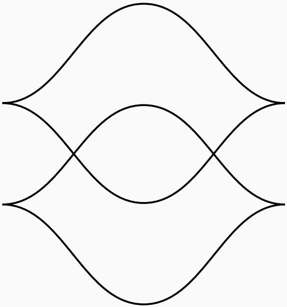
$$\mathbf{R} \times \mathbf{R} \ni (b, \eta) \mapsto \eta^3 - 3(\|b\|^2 - 1)\eta \in \mathbf{R},$$

can be made into a linear-at-infinity generating family f_0^n of Λ_0^n.

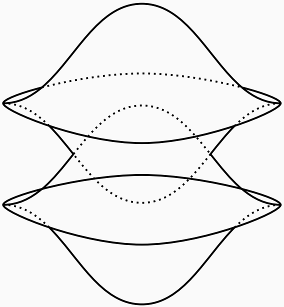


Higher-dimensional Legendrian Hopf links

Let us define $\Lambda^n_{(2)}$ as the Legendrian submanifold of $(J^1\mathbf{R}^n, \xi_{\mathbf{R}^n})$ whose front is two overlapping vertical copies of the front of Λ^n_0.



$\Lambda^1_{(2)} \subset (J^1\mathbf{R}, \xi_{\mathbf{R}})$



$\Lambda^2_{(2)} \subset (J^1\mathbf{R}^2, \xi_{\mathbf{R}^2})$

Generating family $f^n_{//}$ (parallel copy construction)

Step 1.
Start from two copies F^1_1 and F^1_2 of f^n_0, then:

- Decrease slightly the values of F^1_1.
- Shift the fibrewise Morse indices of F^1_2 by $+1$.

Step 2.
Translate F^1_1 and F^n_2 such that $\mathbf{supp}\, F^1_1 \cap \mathbf{supp}\, F^1_2 = \varnothing$.

Step 3.
Iteratively spin $f^1_{//} = F^1_1 + F^2_1$ to construct a GF $f^n_{//}$ of $\Lambda^n_{(2)}$.

Observe that **Step 2** ensures that $f^n_{//}$ has no handleslides.

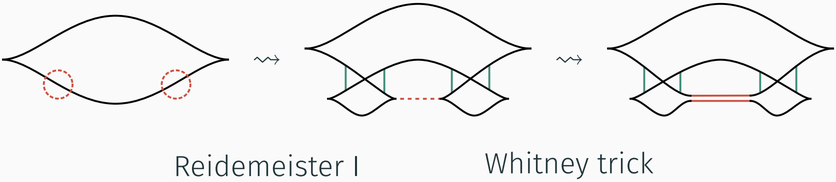
Generating family $f^n_\#$ (surgery construction)

Step 1.

Start from f^n_0 and shift its fibrewise Morse indices by $+1$.

Step 2.

Go through the following steps to construct $f^1_\#$ GF of $\Lambda^1_{(2)}$.

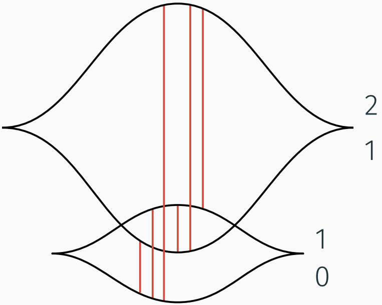


Step 3.

Iteratively spin $f^1_\#$ to construct a GF $f^n_\#$ of $\Lambda^n_{(2)}$.

Generators and grading of the chain complex

Let $(C_\bullet, \partial_\bullet)$ be a generating family chain complex.



Generators. $\leftrightarrow$ Reeb chords.

Grading. $\mu = \Delta\mu_F + \mu_B - 1$.

Perturb $\Lambda^n_{(2)}$ to offset the Reeb chords from the central axis.

Critical point	c_{12}	c_{11}	c_{22}	M_{12}	c_{21}	m_{12}
Grading	$n+1$	n	n	n	$n-1$	0

If $n \geqslant 2$, then: $\partial c_{12} \in \langle c_{11}, c_{22}, M_{12}\rangle$, $\partial M_{12} \in \langle c_{21}\rangle$, $\partial c_{21} = \partial m_{12} = 0$.

Counting gradient staircases

To identify the elements of $e \in \mathcal{M}^{\mathrm{st}}(c_-, c_+)$, observe that

- chord length decreases along e; and
- e is uniquely determined by its vertical fragments;

$\rightsquigarrow$ Examine the strands between which c_- and c_+ lie.

Vertical fragments are made of:

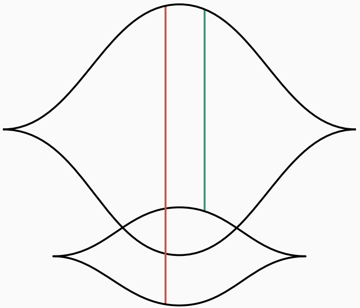
- fibrewise death/birth gradient flow lines;
- fibrewise handleslides; and
- the concatenation of these two types of trajectories.

$\rightsquigarrow$ Jumping between different components requires handleslides.

Simple generating family homology of $f^n_{//}$

Since $f^n_{//}$ has no handleslides, $\partial^n_{//}$ must preserve the indices.

$\rightsquigarrow$ Only $\mathcal{M}^{\mathrm{st}}(c_{12}, M_{12})$ can be nonempty.



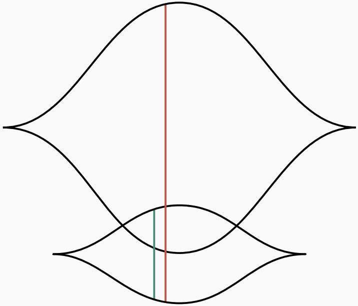
$$\begin{cases} \partial^n_{//} c_{12} = M_{12}, \\ \partial^n_{//} c_{11} = \partial^n_{//} c_{22} = 0, \\ \partial^n_{//} M_{12} = 0. \end{cases}$$

Thus: $\mathrm{GFH}_\bullet(f^n_{//}) = \langle c_{11}, c_{22}, M_{12}, m_{12} \rangle$ and $\Gamma_{f^n_{//}}(t) = 2t^n + t^{n-1} + 1.$

[Proposition 5.1, p. 75, F.]

Simple generating family homology of $f^n_\#$

Since $f^n_\#$ has handleslides, $\partial^n_\#$ no longer preserves the indices.



$$\begin{cases} \partial^n_\# c_{12} = c_{11} + c_{22} + M_{12}, \\ \partial^n_\# c_{11} = \partial^n_\# c_{22} = c_{21}, \\ \partial^n_\# M_{12} = 0. \end{cases}$$

Thus: $\mathrm{GFH}_\bullet(f^n_\#) = \langle M_{12}, m_{12} \rangle$ and $\Gamma_{f^n_\#}(t) = t^n + 1$.

[Proposition 5.2, pp. 75-76, F.]

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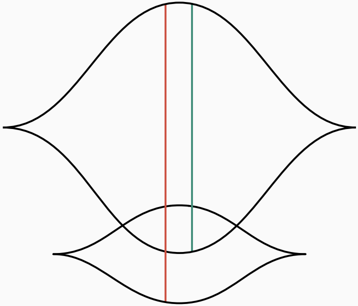

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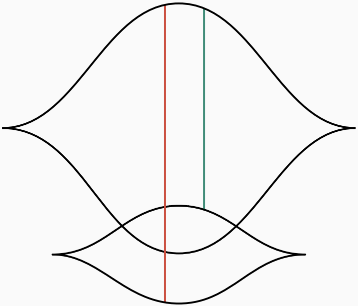

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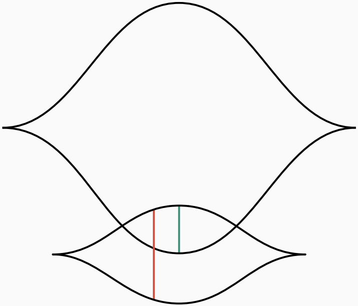

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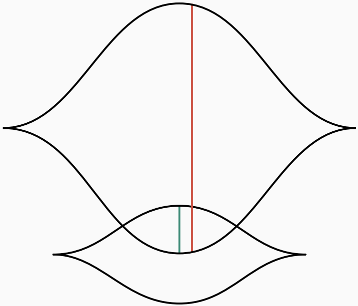

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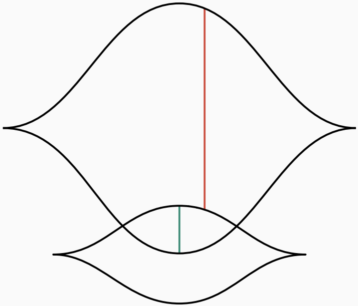

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Thus: $\mathrm{GFH}_\bullet(f^n_\#) = \langle M_{12}, m_{12} \rangle$ and $\Gamma_{f^n_\#}(t) = t^n + 1$.

[Proposition 5.2, pp. 75-76, F.]

Mixed generating family homology of $(f^n_{//}, f^n_{\#})$

Observe that:

- $f^n_{//}$ has no handleslides; and
- $f^n_{\#}$ appears with a minus sign in the difference function;

so that handleslides in gradient staircases can only go upward.

Therefore:

$$\begin{cases} \partial^n_{//,\#} c_{12} = c_{22} + M_{12}, \\ \partial^n_{//,\#} c_{11} = c_{21}, \\ \partial^n_{//,\#} c_{22} = \partial^n_{//,\#} M_{12} = 0, \end{cases}$$

and thus: $\mathrm{GFH}_\bullet(f^n_{//}, f^n_{\#}) = \langle M_{12}, m_{12} \rangle$ and $\Gamma_{f^n_{//}, f^n_{\#}}(t) = t^n + 1$.

[Proposition 5.3, p. 77, F.]

Some research prospects

Gluing conjecture

Showing the one-to-one correspondence between $\mathcal{M}^{\mathrm{st}}(c_-, c_+)$ and $\mathcal{M}(c_-, c_+; g_s)$ now amounts to a gluing theorem.

Conjecture (Gluing, p. 105, F.)

If Λ is generic and $e \in \mathcal{M}^{\mathrm{st}}(c_-, c_+)$ with $|c_-| = |c_+| + 1$, then there exists a unique 1-parameter family $\gamma_s \in \mathcal{M}(c_-, c_+; g_s)$ with $s \to 0$ such that $\gamma_s \xrightarrow[s \to 0]{} e$ in the Floer–Gromov topology.

The proof is expected to rely on the Newton–Raphson method applied to a suitable right-invertible Fredholm operator and a suitable initial smooth approximation of e (pre-gluing).

mGFH is a complete GF invariant

Observe that mGFH fits in the following long exact sequence

$$\cdots \to \mathrm{GFH}_k(f_1, f_2) \xrightarrow{\tau_k} H_k(\Lambda; \mathbb{F}_2) \xrightarrow{\sigma_k} \mathrm{GFH}^{n-k}(f_2, f_1) \xrightarrow{\rho_k} \cdots ,$$

induced by the short exact sequence of the triple $(\delta^{\omega}, \delta^{\varepsilon}, \delta^{-\varepsilon})$ [Bourgeois–Sabloff–Traynor, 2015 & Theorem 2.4, p. 30, F.].

If Λ is connected and $f_1 \sim f_2$, then τ_n is surjective [B.-S.-T., '15 & Theorem 2.2, p. 27, F.] and the converse is conjectured to hold true in all generality.

Conjecture (mGFH is complete, Conjecture 2.1, pp. 56–57, F.)

If Λ is connected, then τ_n is surjective if, and only if, $f_1 \sim f_2$.

Geography questions for GFH and mGFH

The graded vector space geography of GFH is already known [Bourgeois-Sabloff-Traynor, 2015].

Question (Geography of mGFH, Conjecture 2.2, p. 58, F.)

What are the graded vector spaces realised by mGFH*?*

There exists a ring structure on $\mathrm{GFH}^\bullet$ [Myer, 2018].

Question (Ring geography of GFH, p. 106, F.)

What are the possible ring structures on $\mathrm{GFH}^\bullet$*?*

Thank you for your attention!