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RECURSIVE KALMANNET: DEEP LEARNING-AUGMENTED KALMAN FILTER

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About Exail

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- **High-tech French industrial group, with more than 2000 employees in 80 countries. Specialized in:**
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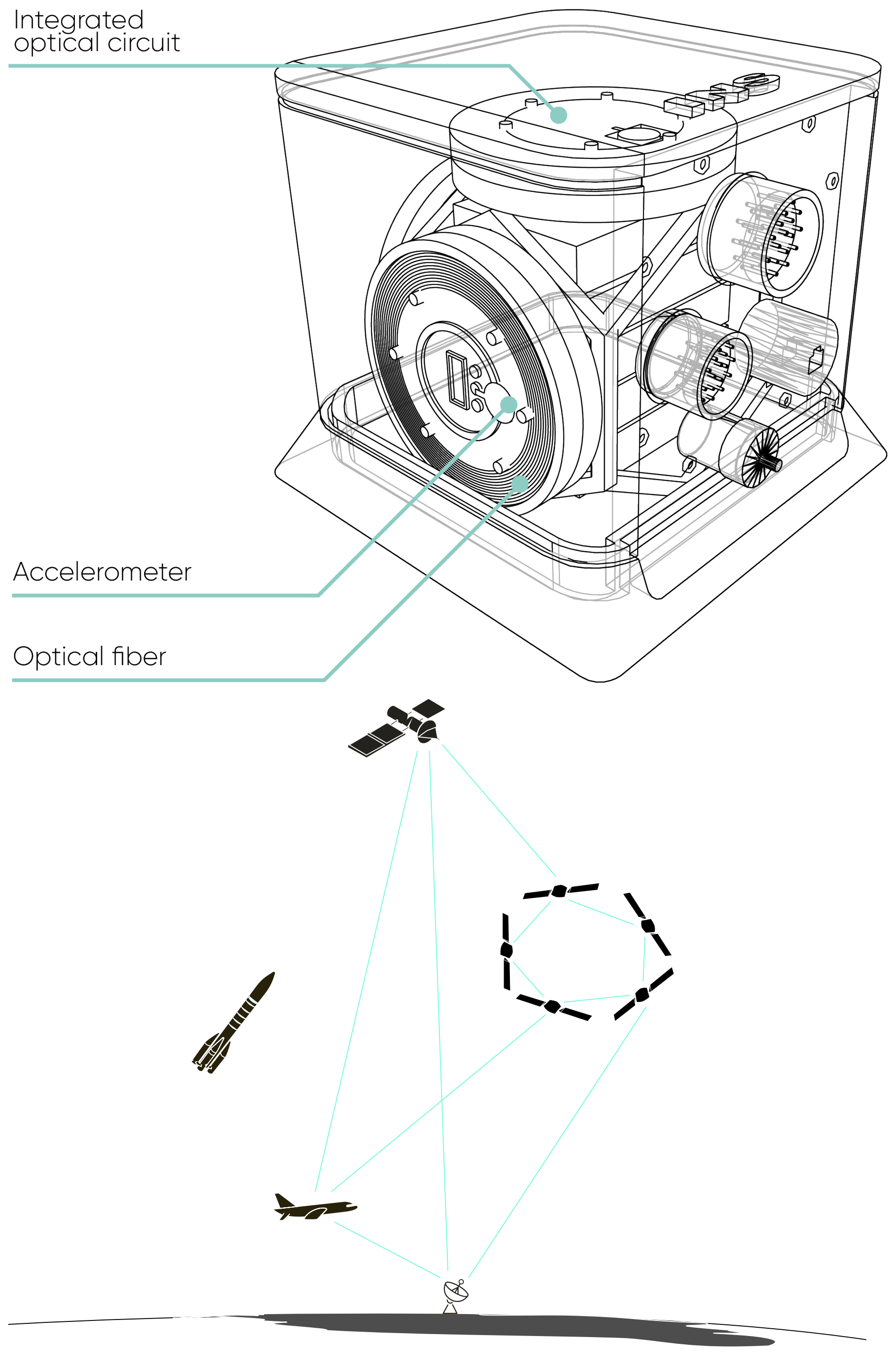
  - Photonics
- **The navigation division produces extremely precise inertial navigation systems:**
  - Based on Fiber-Optic Gyrometers (FOG) and MEMS accelerometers
  - Output position, velocity and attitude
  - Internal and external sensors (*e.g.* GNSS) are fused using **Kalman Filter (KF)** variants



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Problem statement: State and Uncertainty estimation

Linear state-space model:

$$\mathbf{x}_t = \mathbf{F}_t \mathbf{x}_{t-1} + \mathbf{v}_t, \quad \mathbf{v}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_t)$$
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	White	Independant	Gaussian
Best Linear Unbiased Estimator	✗	✗	
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› **Goal: overcome the noise requirements and without the prior knwoledge of $\mathbf{Q}_t$ and $\mathbf{R}_t$**

Deep Learning-Augmented Kalman Filtering methods

- **Goal: address the limitations of the traditional KF**
- **Main idea: learn the Kalman gain by leveraging the estimated states in a supervised fashion**
- **Architecture: a Recurrent Neural Network (RNN) with a set of features such as the innovation**

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  - KalmanNet [1]: pionneer paper tested on model mismatch and system nonlinearities. It lacks the estimation of the error covariance matrix and the tracking of time-varying gains

[1] G. Revach, N. Shlezinger, X. Ni, A. L. Escoriza, R. J. Van Sloun, and Y. C. Eldar, "KalmanNet: Neural Network Aided Kalman Filtering for Partially Known Dynamics," IEEE Trans. Signal Process., vol. 70, pp. 1532–1547, 2022.

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  - Cholesky KalmanNet [3]: add-on split KalmanNet to output the error covariance matrix by calculating its Cholesky factor to ensure its Cholesky factor. Trade-off between state estimation and covariance error matrix accuracy due to the cost function choice

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[3] M. Ko and A. Shafieezadeh, "Cholesky-KalmanNet: Model-Based Deep Learning With Positive Definite Error Covariance Structure," IEEE Signal Process. Lett., vol. 32, pp. 326–330, 2025.

Proposed method: Recursive KalmanNet (RKN)

- **Deep learning model inspired by Kalman filtering, preserving the prediction–correction scheme**
- **Operates without prior knowledge of noise covariance matrices $\mathbf{Q}_t$ and $\mathbf{R}_t$, and without assuming any noise model**
- **Composed of two RNNs:**
  - One dedicated to direct gain estimation
  - Another contributes to error covariance estimation using the generic Joseph's formulation
- **First method to yield accurate state estimation and consistent error covariance thanks to the architecture and tailored cost function**

Kalman Filter

- **Prediction:**

$$\widehat{\mathbf{x}}_{t|t-1} = \mathbf{F}_t \widehat{\mathbf{x}}_{t-1|t-1}$$

$$\mathbf{P}_{t|t-1} = \mathbf{F}_t \mathbf{P}_{t-1|t-1} \mathbf{F}_t^{\top} + \mathbf{Q}_t$$

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- **Correction:**

$$\begin{aligned} \widehat{\mathbf{x}}_{t|t} &= \widehat{\mathbf{x}}_{t|t-1} + \mathbf{K}_t (\mathbf{z}_t - \mathbf{H}_t \widehat{\mathbf{x}}_{t|t-1}) \\ &= (\mathbf{I} - \mathbf{K}_t \mathbf{H}_t) \widehat{\mathbf{x}}_{t|t-1} + \mathbf{K}_t \mathbf{z}_t \end{aligned}$$

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$$= (\mathbf{I} - \mathbf{K}_t \mathbf{H}_t) \mathbf{P}_{t|t-1} \quad \text{[Simplified formulation after plugging-in the gain]}$$

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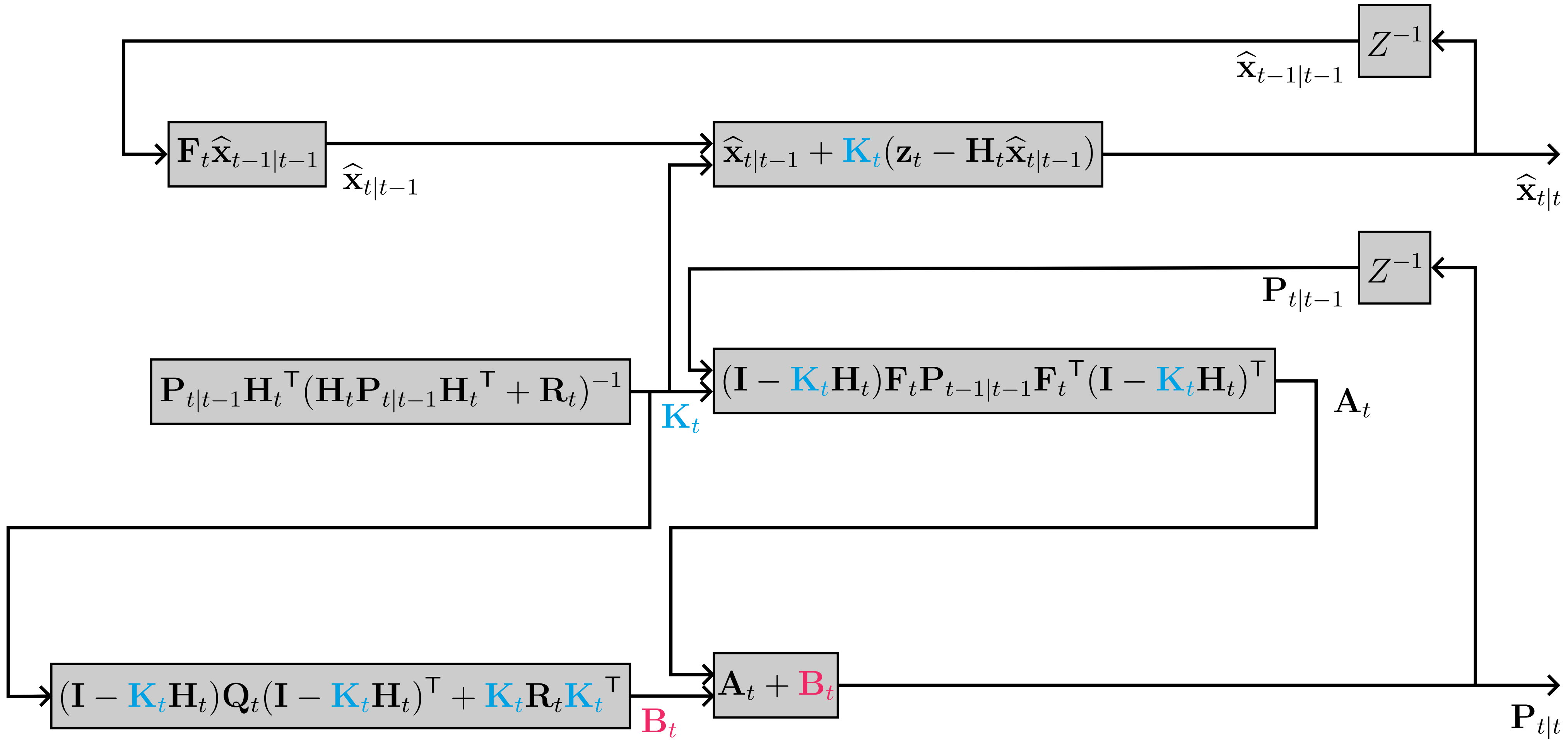


Recursive KalmanNet (RKN) Architecture

- **Prediction:**

  $\widehat{\mathbf{x}}_{t|t-1} = \mathbf{F}_t \widehat{\mathbf{x}}_{t-1|t-1}$

  ~~$\mathbf{P}_{t|t-1} = \mathbf{F}_t \mathbf{P}_{t-1|t-1} \mathbf{F}_t^{\mathsf{T}} + \mathbf{Q}_t$~~

- **Gain:**

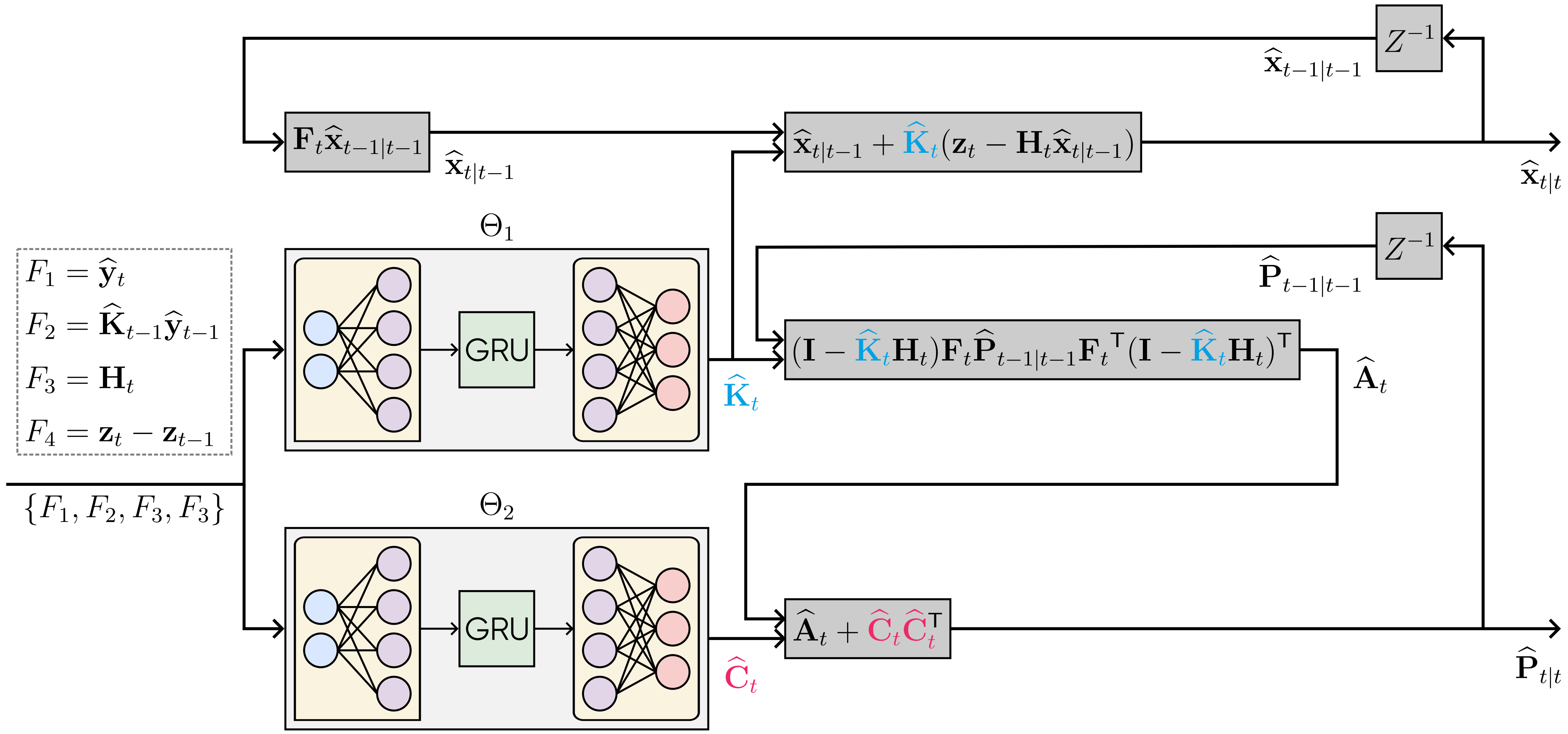
  $\widehat{\mathbf{K}}_t = \mathrm{RNN}_{\Theta_1}(F_1,\ F_2,\ F_3,\ F_4)$

- **Correction:**

  $\widehat{\mathbf{x}}_{t|t} = \widehat{\mathbf{x}}_{t|t-1} + \widehat{\mathbf{K}}_t(\mathbf{z}_t - \mathbf{H}_t \widehat{\mathbf{x}}_{t|t-1})$

  $\widehat{\mathbf{C}}_t = \mathrm{RNN}_{\Theta_2}(F_1,\ F_2,\ F_3,\ F_4)$

  $$\widehat{\mathbf{P}}_{t|t} = \underbrace{(\mathbf{I} - \widehat{\mathbf{K}}_t \mathbf{H}_t)\mathbf{F}_t \widehat{\mathbf{P}}_{t-1|t-1} \mathbf{F}_t^{\mathsf{T}} (\mathbf{I} - \widehat{\mathbf{K}}_t \mathbf{H}_t)^{\mathsf{T}}}_{\widehat{\mathbf{A}}_t} + \underbrace{\widehat{\mathbf{C}}_t \widehat{\mathbf{C}}_t^{\mathsf{T}}}_{\widehat{\mathbf{B}}_t}$$



- **Training is performed by gradient descent on the negative Gaussian log-likelihood of the error estimation:**

$$\mathbf{e}_t^{(i)}(\Theta_1)^{\mathsf{T}} \widehat{\mathbf{P}}_{t|t}^{(i)}(\Theta_1, \Theta_2)^{-1} \mathbf{e}_t^{(i)}(\Theta_1) + \log\det \widehat{\mathbf{P}}_{t|t}^{(i)}(\Theta_1, \Theta_2),$$

averaged over time and batch, where $\mathbf{e}_t^{(i)} = \widehat{\mathbf{x}}_t^{(i)}(\Theta_1) - \mathbf{x}_t$.

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Case study: Heavy-tailed bimodal-Gaussian measurement noise

- 1D constant-speed linear model, with a position measurement

$$
\begin{aligned}
\mathbf{x}_t &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}_{t-1} + \mathbf{v}_t, \quad && \mathbf{x}_t \in \mathbb{R}^2 \\
z_t &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}_t + w_t, && z_t \in \mathbb{R}.
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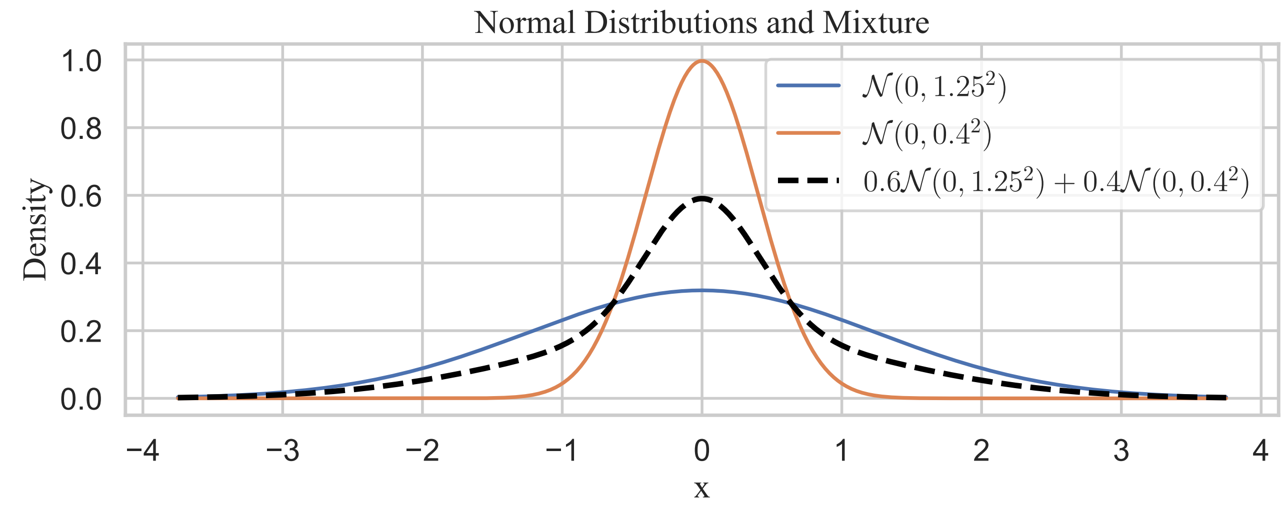
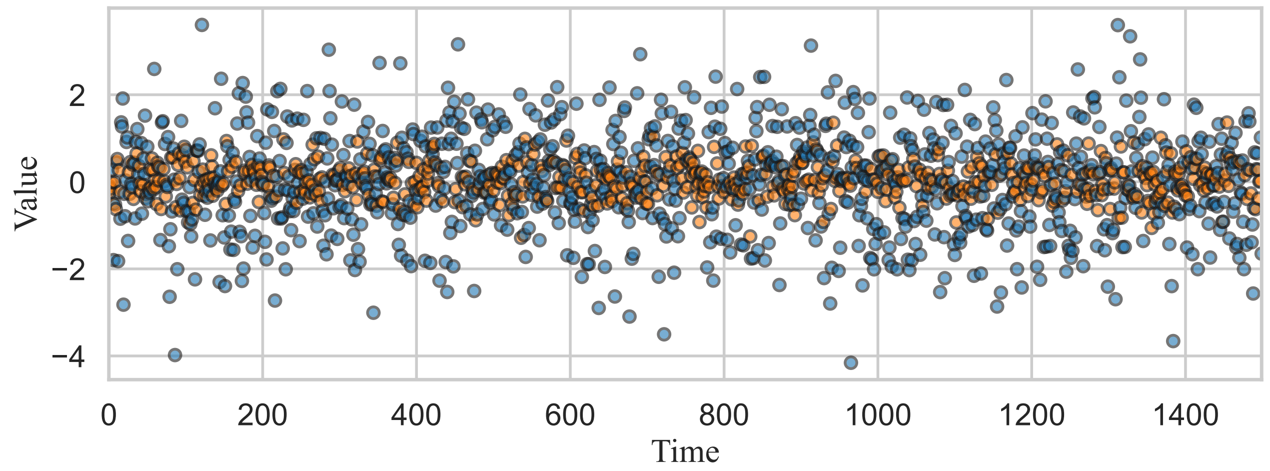
- **The process noise $\mathbf{v}_t$ is zero-mean Gaussian white noise with covariance $\mathbf{Q}_t = \begin{bmatrix} 0 & 0 \\ 0 & {\sigma_v}^2 \end{bmatrix}$**

- **The measurement noise w_t follows a heavy-tailed bimodal-Gaussian distribution:**

$$w_t = Z_t X_t + (1 - Z_t) Y_t$$

  - X_t, Y_t, and Z_t are independent white noise processes
  - X_t and Y_t are Gaussian with variances ${\sigma_1}^2$ and ${\sigma_2}^2$ respectively
  - Z_t is Bernoulli with parameter p
  - w_t is distributed as $p\mathcal{N}(0, {\sigma_1}^2) + (1-p)\mathcal{N}(0, {\sigma_2}^2)$. It has variance $\mathbf{R}_t = Z_t {\sigma_1}^2 + (1 - Z_t){\sigma_2}^2$, and an expected variance of ${\sigma_w}^2 = p{\sigma_1}^2 + (1-p){\sigma_2}^2$





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Results: Performance at varying noise heterogeneity levels

- **Numerical application:** ${\sigma_v}^2 = 10^{-4}, \quad {\sigma_1}^2 = 1.56{\sigma_w}^2, \quad p = 0.6, \quad N = 1000$ (test set size)

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- **Compared methods:**
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  - SO-KF: Sub-Optimal KF with constant $\mathbf{R}_t = {\sigma_w}^2 = p{\sigma_1}^2 + (1 - p){\sigma_2}^2$
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- **Metrics:** $\textbf{MSE} = \frac{1}{T}\sum_{t=1}^{T}\frac{1}{N}\sum_{i=1}^{N}\mathbf{e}_t^{(i)\top}\mathbf{e}_t^{(i)} \quad \textbf{MSMD} = \frac{1}{T}\sum_{t=1}^{T}\underbrace{\frac{1}{N}\sum_{i=1}^{N}\mathbf{e}_t^{(i)\top}{\mathbf{P}_{t|t}^{(i)}}^{-1}\mathbf{e}_t^{(i)}}_{\textbf{CLT}\Rightarrow\sim\mathcal{N}(2,\ 4/N)}$

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  - Cholesky KalmanNet (CKN)
  - Recursive KalmanNet (RKN)
- **Metrics:** $\text{MSE} = \frac{1}{T}\sum_{t=1}^{T}\frac{1}{N}\sum_{i=1}^{N}\mathbf{e}_t^{(i)\mathsf{T}}\mathbf{e}_t^{(i)}$ $\quad \text{MSMD} = \frac{1}{T}\sum_{t=1}^{T}\underbrace{\frac{1}{N}\sum_{i=1}^{N}\mathbf{e}_t^{(i)\mathsf{T}}\mathbf{P}_{t|t}^{(i)-1}\mathbf{e}_t^{(i)}}_{\text{CLT}\Rightarrow\sim\mathcal{N}(2,\,4/N)}$
- **MSE and MSMD for varying noise heteroginity** $\nu = 10\log\frac{\sigma_w{}^2}{\sigma_v{}^2}$

ν	20		30		40		50		60	
	MSE	**MSMD**	**MSE**	**MSMD**	**MSE**	**MSMD**	**MSE**	**MSMD**	**MSE**	**MSMD**
o-KF	-28	2.0	-21	2.0	-14	2.0	-6.9	2.0	0.1	2.0
so-KF	**−26**	2.0	-18	2.0	-11	2.0	-3.9	2.0	3.4	2.0
CKN	-20	4.0	-17	27	-11	3.2	-4.8	6.2	2.4	21
RKN	**−26**	2.0	**−19**	2.0	**−12**	2.0	**−5.1**	1.9	**2.3**	2.2

- **RKN gives closest MSE to O-KF with consistent error covariance reflected by MSMD close to** 2

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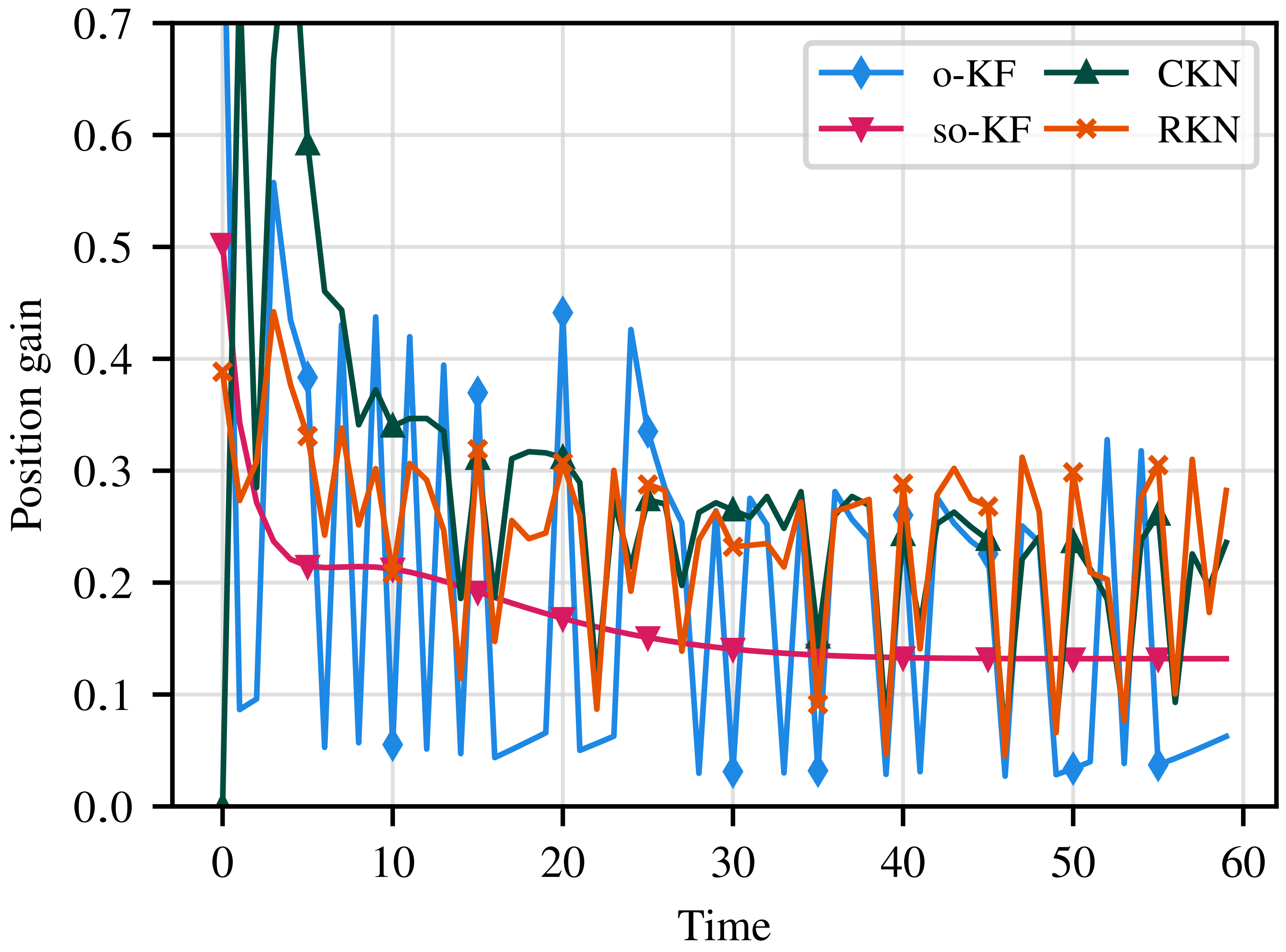
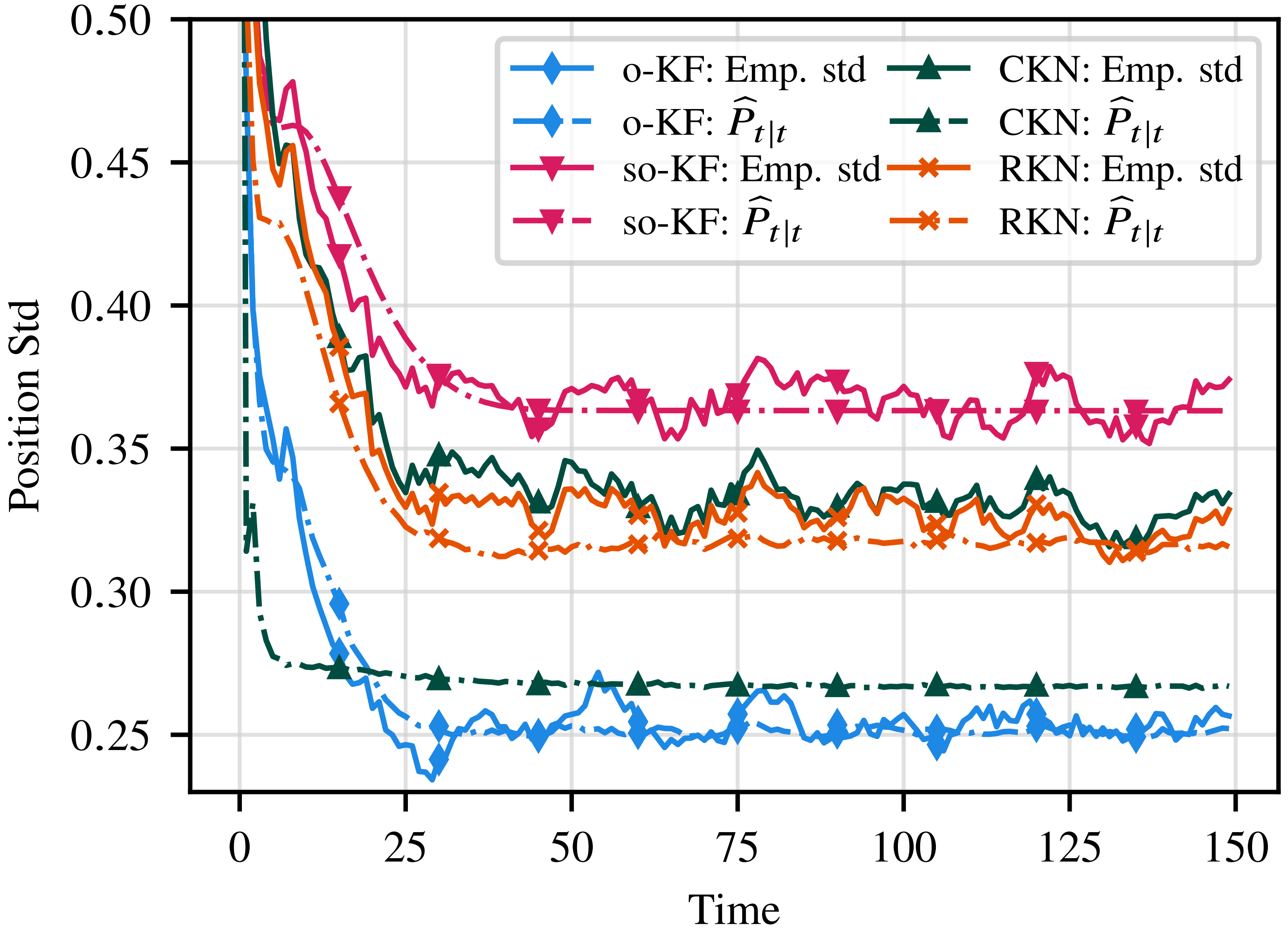
Results: Position gain and error standard deviation estimations

- **fixed** $\nu = 40$ **dB**





- **RKN is able to track challenging time-varying gain behavior**
- **RKN provides consistent position error covariance of the state error and is better with the conventional so-KF**
- **CKN error covariance is not consistent with the empirical error**

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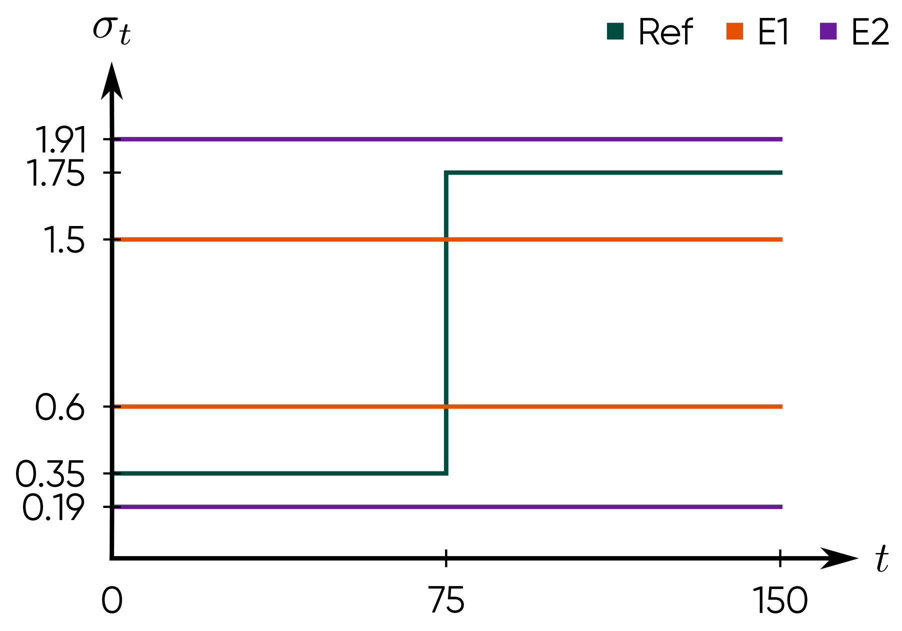
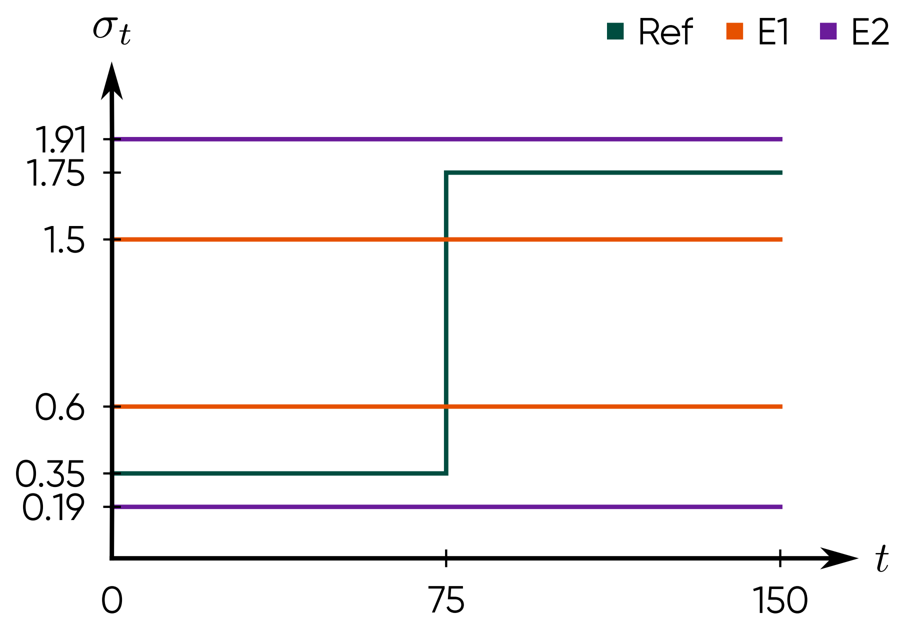
Results: Model generalization capabilities

- $\mathbf{R}_t = {\sigma_t}^2$

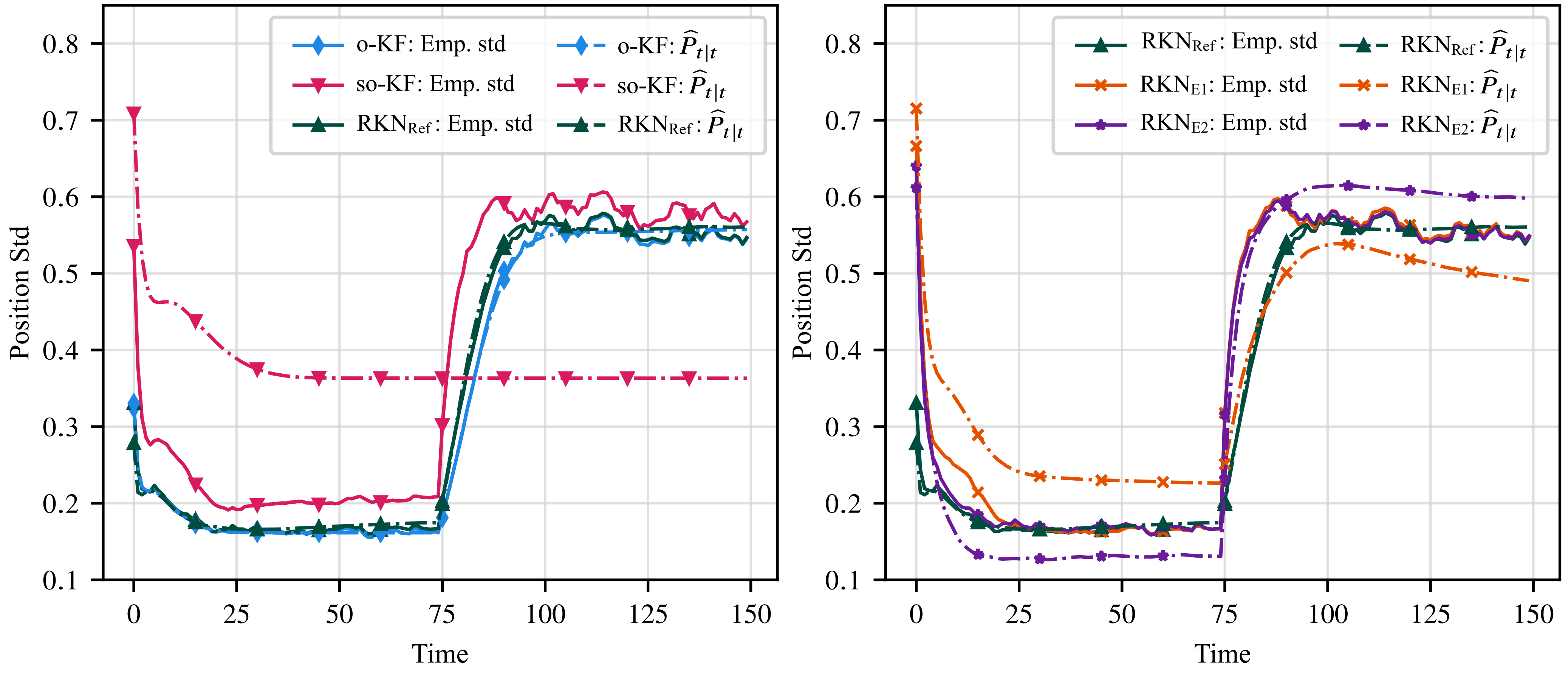


Results: Model generalization capabilities

- $\mathbf{R}_t = {\sigma_t}^2$



- **Training with REF, E1 or E2. Test with REF**



- **No degradation in estimation precision.**
- **Loss of uncertainty consistency in error covariance estimation.**

[4] C. Falcon, H. Mortada, M. Clavaud, and J.-P. Michel, "Recursive KalmanNet : Analyse des capacités de généralisation d'un réseau de neurones récurrent guidé par un filtre de Kalman," in 30e Colloque sur le traitement du signal et des images. GRETSI, 2025.

Conclusion

- RKN, a deep-learning augmented Kalman filter that leverages two recurrent neural networks to estimate gain and error covariance using a formulation derived from Joseph's equation
- In scenarios with bimodal Gaussian noise, RKN outperformes conventional Kalman filters and a state-of-the-art deep learning approach, delivering accurate state and covariance estimates that closely reflect actual errors
- RKN shows strong potential in cases where classical Kalman filtering falls short, such as in non-linear systems or with incomplete state-space models

Thank you for your attention!

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