REPORT ON DISSERTATION OF CYRIL FALCON:
GENERATING FAMILY HOMOLOGIES OF LEGENDRIAN SUBMANIFOLDS AND MODULI SPACES OF GRADIENT STAIRCASES

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The dissertation of Cyril Falcon establishes important and difficult analytical results that support a conjecture made about a decade ago by Henry and Rutherford regarding how one can calculate the boundary map for generating family homology of a Legendrian submanifold via the Legendrian's front projection. This research also leads to a number of interesting new conjectures and promising avenues for future research. I believe Falcon should be given authorization to defend his thesis.

Legendrian submanifolds of a contact manifold are important objects in symplectic and contact topology. The 1-jet bundle, J^1B, has a canonical contact structure, and the techniques of generating families (of functions) and holomorphic curves have been used to establish non-classical invariants of Legendrian submanifolds of J^1B. In particular, it is possible to define generating family homology groups, $GFH_*(\Lambda, f)$, of a Legendrian submanifold Λ when the Legendrian is equipped with a generating family f, and linearized contact homomology groups, $LCH_*(\Lambda, \epsilon)$, when the Legendrian is equipped with an augmentation ϵ of its DGA, which is defined through the theory of holomorphic curves. It is known that for 1-dimensional Legendrians, the existence of a generating family f implies the existence of an augmentation ϵ_f such that $GFH_*(\Lambda, f) \cong LCH_*(\Lambda, \epsilon_f; \mathbb{F}_2)$; in general there are many intriguing parallels between results established through the techniques of generating families and holomorphic curves/augmentations. The construction of generating family homology has the benefit that it is founded on classical Morse theory. The construction of contact homology is more difficult as it requires infinite-dimensional analysis, however a benefit of linearized contact homology is that there exist combinatorial ways to compute the boundary map directly from the front projection of the Legendrian. In contrast, for generating family homology, the boundary map is defined by the gradient trajectories of the "difference function" of the generating family, $\delta_f : B \times \mathbb{R}^{2N} \to \mathbb{R}$, for potentially large N; although the critical points of δ_f can easily be identified on the front projection of Λ, the trajectories live in $B \times \mathbb{R}^{2N}$ and need not have a representation on the front projection. Rather than working in the front projection, previous calculations of generating family homology have typically resulted from algebraic topology techniques, such as long exact sequences, or, for 1-dimensional Legendrians, by going through the generating family to augmentation correspondence and doing the combinatorial calculations for linearized contact homology.

Approximately a decade ago, Henry and Rutherford made an important conjecture about how one could calculate the boundary map for generating family homology, $GFH_*(\Lambda, f)$, from a front diagram of the Legendrian submanifold Λ that has been decorated with handle-slide information from f. Henry and Rutherford's idea is to take advantage of the choice of metric and consider a limiting family of metrics where one speeds up the fiberwise gradient flow. Falcon has put this idea into practice. Given a metric $g = g_B \oplus g_F$, Falcon has found that there are analytical advantages to instead slow down the base component of the gradient flow by considering the family

$$g_s = \frac{g_B}{s} \oplus g_F, \quad s \in (0, 1].$$

As Falcon points out in Chapter 3, Equations (5)-(7), the gradient trajectories $(b_s(t), \eta_s(t)) \in B \times \mathbb{R}^{2N}$ satisfy

$$\begin{aligned} \partial_t b_s(t) &= -s\nabla_{g_B}\delta(b_s(t), \eta_s(t)), \\ \partial_t \eta_s(t) &= -\nabla_{g_F}\delta(b_s(t), \eta_s(t)), \end{aligned} \tag{5}$$

which when written with respect to the "slow time" reparameterization $\tau = st$ become

$$\begin{aligned} \partial_\tau b_s(\tau) &= -\nabla_{g_B}\delta(b_s(\tau), \eta_s(\tau)), \\ s\partial_\tau \eta_s(\tau) &= -\nabla_{g_F}\delta(b_s(\tau), \eta_s(\tau)). \end{aligned} \tag{6}$$

As long as $s \neq 0$, systems (5) and (6) are equivalent, however in the limit as $s \to 0$, system (5) converges to the "vertical" flow lines

$$\begin{aligned} \partial_t b_s(t) &= \mathbf{0}, \\ \partial_t \eta_s(t) &= -\nabla_{g_F}\delta(b_s(t), \eta_s(t)), \end{aligned} \tag{7}$$

while in the limit as $s \to 0$, system (6) converges to the "horizontal" flow lines

$$\begin{aligned} \partial_\tau b_s(\tau) &= -\nabla_{g_B}\delta(b_s(\tau), \eta_s(\tau)), \\ \mathbf{0} &= -\nabla_{g_F}\delta(b_s(\tau), \eta_s(\tau)). \end{aligned} \tag{8}$$

Given positively valued critical points $c_\pm$ and a sequence of $(s_k) \to 0$, $s_k \in (0, 1]$, Falcon defines (**Definition 3.1**) an **HR-sequence** γ_k to be a sequence of trajectories in the unparameterized moduli spaces $\mathcal{M}(c_-, c_+; g_{s_k}, \delta)$. Motivated by Equations (7), (8), in **Definition 3.2** Falcon defines a set of **gradient staircases** $\mathcal{M}^{st}(c_-, c_+; g, \delta)$ made from "horizontal" steps that are obtained by flowing along the sheets of the front projection of Λ, and "vertical" risers that can be represented by vertical line segments of the front projection that connect sheets of the front projection. Falcon then states **Conjecture A** (more precisely stated as **Conjecture 3.1**), which is motivated by one of Henry-Rutherford but is more precise in details, that if Λ is gradient generic (Definition 1.11), then if $s_0 \in (0, 1]$ is sufficently small, there exists a 1-1 correspondence between the unparameterized moduli space $\mathcal{M}(c_-, c_+; \delta, g_{s_0})$ and the unparameterized moduli space $\mathcal{M}^{st}(c_-, c_+; \delta, g_1)$. A proof of this conjecture of this sort generally involves two directions: "compactness" and "gluing". In this dissertation, Falcon's **Theorem A (Theorem 4.1)** establishes the compactness portion: Legendrian submanifolds can be perturbed in such a way that any HR-sequence will have a subsequence that converges to a gradient staircase in the Floer-Gromov topology. In **Theorem B (Theorem 4.3)** Falcon establishes that it is necessary to perturb to gradient generic Legendrians, which in **Definition 1.11** are defined to be those with bounded tangency of the gradient vector field to the submanifold corresponding to the front of the Legendrian, in order to avoid infinite staircases. As proved in **Theorem C (Theorem 1.2)**, the set of gradient generic Legendrians is open and dense in the set of all Legendrian embeddings. As explained in Chapter 1, Falcon restricts to the Legendrians whose projection to the base B have only Whitney pleat singularities in order to avoid the full Thom-Boardman hierarchy and simplify some of the arguments. This hypothesis could likely be removed but Legendrians with Whitney pleat singularities is a robust class that includes all 1-dimensional Legendrians and the commonly examined examples in higher dimensions.

The most technically challenging arguments occur in Chapter 4 where Falcon proves Theorem A (Theorem 4.1). Given an HR-sequence, the vertical fragments of the limit are recovered through an application of the Arzelà-Ascoli Theorem. However, recovering the horizontal fragments of the limit is much more challenging: the changing speed parameter destroys the uniform C^2-bounds required to apply Arzelà-Ascoli. To recover the limiting horizontal fragments, Falcon makes a careful quantitative analysis of the negative gradient flow dynamics of the difference function in a region of $B \times \mathbb{R}^{2N}$ corresponding to a neighborhood of the front of the Legendrian. In particular, upper bounds from Propositions 4.2-4.5 allow him to prove in Theorem 4.2 uniform convergence on an interval from pointwise convergence on the interval's boundary. The analysis and resulting estimates are slightly different depending on whether or not one is working in a neighborhood of a region corresponding to a cusp of the Legendrian front. Some of these estimates are similar to yet different from those found in work of Bourgeois-Oancea: in this work, a new "linear" term arises from the changing time parameter s. Falcon does an excellent job at carefully justifying his estimates and writing remarks to explain the statements and computations in these technical results. He also gives some examples to show the sharpness of some of the estimates.

Although generating family homology and linearized contact homology were first defined via a single generating family and a single augmentation, there naturally exist mixed generating family homology, $GFH_*(\Lambda, f_1, f_2)$, which is calculated using the difference function of two generating families, and there exists bilinearized Legendrian contact homology, $LCH_*(\Lambda, \epsilon_1, \epsilon_2)$, which is calculated with the input of two augmentations. In Chapter 2, Falcon establishes a duality long exact sequence for mixed generating family homology, **Theorem 2.3**, that generalizes the duality exact sequence in the non-mixed setting established by myself, Bourgeois, and Sabloff. However, there is an interesting difference in the mixed version: now a particular map $\tau_n : GFH_n(\Lambda, f_1, f_2) \to H_n(\Lambda, \mathbb{F}_2)$ need not be surjective. This leads to the interesting and compelling **Conjecture 2.1**, that two generating families are equivalent if and only if τ_n is surjective. Here equivalence of generating families is with respect to fiber-preserving diffeomorphism and stabilization. An analogous statement with respect to bilinearized contact homology has been proved by Bourgeois-Gallant, where equivalence of augmentations is given by DGA-homotopy. Establishing whether or not Conjecture 2.1 is true would be important for further understanding the relationship between generating families and augments. This conjecture is listed among the prospective future projects.

It is convenient to encode the ranks of the Legendrian generating family/linearized contact homology groups as the coefficients in a polynomial $\Gamma(t)$, and the "Geography Problem" asks what polynomials can be realized. The non-mixed version of the duality exact sequence was instrumental in defining "GF-admissable polynomials", which are polynomials satisfying certain symmetries, that could be realized by $GFH_*(\Lambda, f)$ as $*$ varies. Falcon's **Conjecture 2.2** defines a notion of mixed generating family admissable polynomials and conjectures the mixed geography that can be realized. Again, this conjecture has a parallel result for bilinearized contact homology, established by Bourgeois-Gallant. Establishing whether or not Conjecture 2.2 is true would be important for further understanding the relationship between generating families and augments and is listed among the prospective future projects.

The "gluing portion" of Conjecture A (Conjecture 3.1) is another of the worthy projects listed in Falcon's research prospects. The gradient staircases and the gradient flow trees, as defined by Ekholm, would offer a path to understand in all dimensions, the relationship between generating family homology and linearized contact homology. Assuming Conjecture A holds, in Chapter 5 Falcon shows how the gradient staircases can be used to make homological computations with generating families of some generic Legendrians. In particular, he calculates the gradient staircases for two generating families $f_{\|}$ and $f_{\sharp}$ for the n-dimensional Legendrian Hopf link and for generating families $F_{\|}, F_{\sharp}$ for a connected Legendrian obtained by a connect sum of these Hopf links; in dimension 1, this connected Legendrian is a positive Legendrian trefoil. Calculations of $GFH_*(\Lambda, f_{\|})$ and $GFH_*(\Lambda, f_{\sharp})$ via the conjectured correspondence with gradient staircases show that the $f_{\|}$ and $f_{\sharp}$ are not equivalent, a result that we currently have no other way to tackle in arbitrary dimensions. While for the connected Legendrian, the gradient staircase calculations show that $GFH_*(\Lambda, F_{\|}) \cong GFH_*(\Lambda, F_{\sharp})$, the mixed version would imply, assuming Conjecture 2.1, that $F_{\|}$ and $F_{\sharp}$ are not equivalent. These calculations of the Hopf link and the connected Legendrian are likely to be crucial in verifying parts of Conjecture 2.2 about realizing mixed generating family geography.

For the reasons I describe above, I believe that the dissertation of Cyril Falcon establishes important results in the area of contact and symplectic topology. The dissertation is well written with careful attention to details, includes many original ideas, and incorporates an excellent literature review that puts his results in a broader perspective. I had some minor revisions that I have already sent to Falcon. I believe Falcon should be given authorization to defend the thesis.

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